Nonlinear Viscoelastic Solids—A Review

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Abstract: Elastomers and soft biological tissues can undergo large deformations and exhibit time dependent behavior that is characteristic of nonlinear viscoelastic solids. This article is intended to provide an overview of the subject of nonlinear viscoelastic solids for researchers who are interested in studying the mechanics of these materials. The article begins with a review of topics from linear viscoelasticity that are pertinent to the understanding of nonlinear viscoelastic behavior. It then discusses the topics that enter into the formulation of constitutive equations for isotropic, transversely isotropic and orthotropic nonlinear viscoelastic solids. A number of specific forms of constitutive equations have been proposed in the literature and these are discussed. Attention is restricted to constitutive equations that are phenomenological rather than molecular in origin. The emphasis is then on nonlinear single integral finite linear viscoelastic and Pipkin–Rogers constitutive equations, the latter containing the quasi-linear viscoelastic model used in biomechanics of soft tissue. Expressions for the Pipkin–Rogers model are provided for isotropy, transverse isotropy and orthotropy.

The constitutive equations are then applied to the description of homogeneous triaxial stretch and simple shear histories. The special case of uniaxial stretch histories is analyzed in detail. There is a discussion of the deviation from linear behavior as nonlinear effects become important. Non-homogeneous deformations are considered next. The combined tension and torsion of a solid cylinder on an incompressible, isotropic nonlinear viscoelastic solid is discussed in detail because of its importance in experiments involving viscoelastic materials. A large number of solutions to boundary value problems have appeared in the literature and many of these are summarized. The article concludes with comments about interesting topics for further research.

Key Words: Nonlinear single integral constitutive equations, Volterra integral equations, membranes, uniaxial and tension-torsion histories

1. INTRODUCTION

Plastics, rubber, asphalt and biological materials exhibit time dependent mechanical response such as creep under constant load, stress relaxation under constant deformation and delayed strain recovery on unloading. This time dependence plays an important role in their performance when used in load-bearing applications. For example, the time dependence of plastics, rubber and asphalt is related to their dimensional stability following processing, damping properties in vibration isolation and noise abatement, and heating during cyclic loading. The time dependence of biological materials is a factor in understanding their function and performance.

The theory of nonlinear viscoelastic solids provides a framework for modeling the time and deformation dependent phenomena exhibited by these materials. There are two important

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aspects to this modeling. The first is the development of a constitutive equation that can accurately describe the mechanical response of a viscoelastic material. The second is the development of methods for using constitutive equations in conjunction with the governing equations of thermo-mechanics to determine stresses and deformations in structures made of these materials.

Although the theory of nonlinear viscoelasticity was formulated about 40 years ago, its use in engineering applications has been limited. The books by Findley et al. [1] and Lockett [2] represent the state of development of the subject as of about 30 years ago. Findley et al. discussed a constitutive theory that had received a lot attention by contemporary researchers and some of its structural applications. However, the theory fell into disuse because of its practical limitations. Lockett [2] provided a summary of the contemporary constitutive theories, but few applications. In a later review article, Morman [3] described the status of nonlinear viscoelasticity as applied to rubbery materials and suggested important promising directions for further development. The emphasis was again on constitutive equations. Schapery [4] reviewed constitutive theories for fracture and strength of nonlinear viscoelastic solids, while Drapaca et al. [5] reviewed mathematical issues underlying the formulation of constitutive equations for nonlinear viscoelastic solids.

The past few years have seen an increasing interest in the subject of nonlinear viscoelastic solids as a result of research in the biomechanics of soft tissue and the need for efficient engineering of polymeric and elastomeric structural components. It is therefore useful to provide of an overview of the current state of the subject. This article presents an introduction to the continuum theory of nonlinear viscoelastic solids, discussing both constitutive equations and solutions to boundary value problems that have appeared in the literature.

The outline of this paper is as follows. Section 2 presents a summary of results from linear viscoelasticity. This serves two purposes. First, the essence of viscoelasticity is the time dependence of the mechanical response. In the context of linear viscoelasticity, the essential characteristics of this time dependence can be discussed without considering complications with the size of the deformation. Second, these characteristics will be used in Section 14 to provide a basis of comparison for recognizing the onset of nonlinear response. The kinematics of solids undergoing large deformations and the laws of continuum thermo-mechanics are presented, respectively, in Sections 3 and 4. The constitutive equation for linear viscoelasticity presented in Section 2 illustrates the notion that the materials under consideration are materials with a memory, that is, the current stress depends on the previous deformation history. The constitutive assumption expressing this notion for nonlinear viscoelastic materials is introduced in Section 5, along with restrictions arising from the consideration of superposed rigid body rotations, material symmetry and the assumption of incompressibility. The constitutive equations presented in this article are phenomenological rather than molecular in origin. That is, the form of the constitutive equation arises from mathematical assumptions about the mechanical response rather than assumptions about molecular mechanisms underlying that response. Several phenomenological types of constitutive equations have been proposed in the literature and these are summarized in Section 6. The method of determining material symmetry restrictions on these proposed constitutive equations is described in Section 7. Results for isotropic materials are stated in Section 8. Results for transverse isotropy and orthotropy for a specific constitutive equation are stated in Section 9. An important concept in viscoelasticity is that of a clock, which accounts for time dependent material processes that are affected by temperature or deformation. This is discussed in Section 10.

The remainder of the article is concerned with applications of the constitutive equation. The constitutive equations emphasized in this article are expressed in terms of integrals over the deformation history. Applications involving these constitutive equations lead to Volterra integral equations, which are routinely solved by numerical methods. The essential features of numerical methods of solution are described in Section 11. Section 12 describes the predictions of two specific constitutive equations when a material undergoes a step change in a general homogeneous deformation. Section 13 considers the particular homogeneous deformation where a block undergoes triaxial stretch histories. Expressions relating stress and stretch are developed for the two specific constitutive equations of interest. Section 14 contains a detailed discussion of uniaxial stretch histories. It points out the connection between linear and nonlinear viscoelasticity, including the small strain limit, the forms of expressions for stretch dependent stress relaxation, the recognition of the onset of nonlinearity using constant stretch rate histories or sinusoidal oscillations and small deformations superposed on finite uniaxial stretch. Constitutive expressions for two other homogeneous deformations are given, biaxial extension histories in Section 15 and simple shear histories in Section 16. There are several families of non-homogeneous deformations that are possible in any incompressible, isotropic solid. These are presented in Section 17. One of these deformations can be used to describe the combined tension and torsion of a solid circular cylinder. Section 18 contains a detailed discussion of this problem for two reasons. First, it provides an example of the use of a constitutive equation for nonlinear viscoelasticity in solving a boundary value problem. Second, this important deformation is used in experimental work. Expressions relating axial force, twisting moment, axial stretch and twist are developed for one of the constitutive equations and can be used in interpreting experimental results. The solutions to many boundary value problems involving large deformation of nonlinear viscoelastic solids have appeared in the technical literature. A non-exhaustive summary showing the breadth of applications is provided in Section 19, probably the first such summary. This article concludes in Section 20 with suggested directions for further research in nonlinear viscoelastic solids.

2. SOME RESULTS FROM LINEAR VISCOELASTICITY

As the word “viscoelasticity” suggests, the type of mechanical response of interest in this article involves aspects of the response of elastic solids and of viscous fluids. Since the response of fluids involves flow, or continuing deformation as time increases, it is necessary in discussing viscoelastic response to account explicitly for time as a physical parameter. A detailed comparison of viscoelastic response with that of elastic solids and viscous fluids that shows the importance of time as a physical parameter is presented in [6, pp 1–10].

In order to recognize that a material is exhibiting nonlinear viscoelastic response, it is useful to begin by presenting a number of results from linear viscoelasticity. Each result from linear viscoelasticity presented here will be contrasted in a later section with the corresponding result when there is nonlinear viscoelastic response. These results are presented in the context of one-dimensional stress and strain states, the material being either in uniaxial extension or simple shear. Thus, let σ denote either a normal or shear stress and ε denote the corresponding normal or shear strain. Attention is confined to conditions when the material is initially undeformed and unstressed, that is $\varepsilon(t) = 0$ and $\sigma(t) = 0$ at times $t < 0$. Because the mechanical response of a viscoelastic solid depends on the history of its deformation, it is important to distinguish between the current time "t" and a generic earlier time "s", $s \in [0, t]$. The terminology "stress history" or "strain history" is used to refer to the set of values for the stress $\sigma(s)$ or the strain $\varepsilon(s)$, respectively, for $s \in [0, t]$.

2.1. Creep

Let a specimen be subjected to a step stress history, in which the stress is instantaneously increased to some value σ_o at $t = 0$ and then held fixed. The typical strain response consists of (i) an instantaneous increase in strain at $t = 0$ followed by (ii) continued straining in time at a non-constant rate and (iii) an asymptotic approach to some limit value at time increases. The behavior is called *creep*.

Let $J(t, \sigma_o)$ denote the strain at time t when the value of the stress is fixed at σ_o. Then, (i) $J(t, \sigma_o) = 0$ when $t < 0$, (ii) $J(t, \sigma_o)$ jumps to the value $J(0, \sigma_o)$ at $t = 0$, and (iii) $J(t, \sigma_o)$ monotonically increases to the limit value denoted by $J(\infty, \sigma_o)$ as $t \to \infty$. The jump in strain $J(0, \sigma_o)$ at $t = 0$ indicates instantaneous springiness or elasticity. The fact that the material reaches a non-zero limit value of strain indicates solid behavior. If the strain were to increase without bound, it would indicate fluid behavior, which is not considered here. The relations σ_o vs. $J(0, \sigma_o)$ and σ_o vs. $J(\infty, \sigma_o)$ describe, respectively, instantaneous elastic response and the long-time or equilibrium elastic response. $J(t, \sigma_o)$ has a different dependence on time t and stress σ_o for each material, and is therefore considered to be material property called the *creep function*.

2.2. Stress Relaxation

Let a specimen be subjected to a step strain history, in which the strain is instantaneously increased to some value ε_o at $t = 0$ and then held fixed. The typical stress history required to produce this strain history consists of (i) an instantaneous increase in stress at $t = 0$ followed by (ii) a gradual monotonic decrease of stress at a non-constant rate and (iii) an asymptotic approach to some non-zero limit value as time increases. The behavior is called *stress relaxation*.

Let $G(t, \varepsilon_o)$ denote the stress at time t when the value of the strain is fixed at ε_o. Then, (i) $G(t, \varepsilon_o) = 0$ when $t < 0$, (ii) $G(t, \varepsilon_o)$ jumps to the value $G(0, \varepsilon_o)$ at $t = 0$, and (iii) $G(t, \varepsilon_o)$ monotonically decreases to the non-zero limit value denoted by $G(\infty, \varepsilon_o)$ as $t \to \infty$. The jump in stress $G(0, \varepsilon_o)$ at $t = 0$ is another indication of instantaneous springiness or elasticity. That fact that a non-zero stress $G(\infty, \varepsilon_o)$ is required to maintain the strain at ε_o is another indication that the material is a solid. If $G(\infty, \varepsilon_o) = 0$, then no stress would be required to hold the material in a strained state, a characteristic of the response of fluids. The relations $G(0, \varepsilon_o)$ vs. ε_o and $G(\infty, \varepsilon_o)$ vs. ε_o also describe, respectively, instantaneous elastic response and the long-time or equilibrium elastic response. $G(t, \varepsilon_o)$ has a different dependence on time t and strain ε_o for each material, and is therefore considered a material property called the *stress relaxation function*.

2.3. Isochrones

The creep and stress relaxation functions have two independent variables, time and stress or strain, respectively. Their dependence on time can be determined from creep or stress relaxation experiments. Isochrones are useful for determining their dependence on stress or strain. Suppose a program of stress relaxation experiments is carried out for different values of ε_o. At a fixed time $\hat{t}$, the plot of $\sigma(\hat{t}) = G(\hat{t}, \varepsilon_o)$ vs. ε_o is called a *stress relaxation isochrone*. In a similar manner, consider creep experiments carried out for different values of σ_o. At a fixed time $\hat{t}$, the plot of σ_o vs. $\varepsilon(\hat{t}) = J(\hat{t}, \sigma_o)$ is called a *creep isochrone*.

2.4. Constitutive Assumption

The phenomena of creep and stress relaxation show that the mechanical response of the material depends on time. They also illustrate the duality of responses, strain is found under stress control conditions, or stress is found under strain control conditions. This further raises the question of how to determine the strain response when the stress varies with time or the stress response when the strain varies with time. Experimental results provide further evidence of this time dependence and imply that the stress $\sigma(t)$ at time t depends on the preceding strain history, $\varepsilon(s)$, $s \in (-\infty, t]$, or that that the strain $\varepsilon(t)$ at time t depends on the preceding stress history, $\sigma(s)$, $s \in (-\infty, t]$. It was assumed earlier that $\sigma(s) = \varepsilon(s) = 0$, $s \in (-\infty, 0)$. The notation $s \in (-\infty, t]$ is used for convenience and to allow for a history to have a jump from the value of zero at $t = 0-$ to a non-zero value at $t = 0+$.

The constitutive assumption for the stress at time t in terms of the strain history up to time t is denoted by

$$\sigma(t) = \hat{\mathcal{G}}\left[\varepsilon(s)|_{s=-\infty}^{t}; t\right]. \tag{2.1}$$

$\hat{\mathcal{G}}$ is called a *response functional*. The notation $\varepsilon(s)|_{s=-\infty}^{t}$ indicates dependence on the entire strain history, including the jump at $t = 0$. The explicit dependence on t indicates that the material is aging, i.e. when there is curing of concrete or epoxy. The explicit dependence on the parameter t then represents the time since the material was created. Equation (2.1) represents the essential nature of viscoelasticity, the evolution of a strain history as time increases determines the evolution of the corresponding stress history as time increases.

There can also be the dual constitutive assumption for the strain in terms of stress history,

$$\varepsilon(t) = \hat{\mathcal{J}}\left[\sigma(s)|_{s=-\infty}^{t}; t\right]. \tag{2.2}$$

This shows the *dual* nature of viscoelasticity. For every statement of stress in terms of strain history, there is a dual statement of strain in terms of stress history.

2.5. Linearity

A central issue in the modeling of viscoelastic materials is the determination of the mathematical form of the response functional $\hat{\mathcal{G}}$ in (2.1) or $\hat{\mathcal{J}}$ in (2.2). An important assumption in this regard is that the material response is *linear*. The property of linearity of response consists of two conditions: scaling and superposition. These are discussed here only for the stress response to a strain history. Analogous comments apply to the strain response to a stress history.

Let $\varepsilon_1(s)$, $\varepsilon_2(s)$, $s \in (-\infty, t]$ be two strain histories whose stresses at time t are $\sigma_1(t)$, $\sigma_2(t)$, respectively. Let λ_1, λ_2 be constants. The composite strain history

$$\varepsilon(s) = \lambda_1\varepsilon_1(s) + \lambda_2\varepsilon_2(s), \quad s \in (-\infty, t], \tag{2.3}$$

is constructed by scaling strain history $\varepsilon_1(s)$ by λ_1, $\varepsilon_2(s)$ by λ_2 and superposing (or adding) the results. If the stress for the strain history (2.3) is given by

$$\sigma(t) = \lambda_1\sigma_1(t) + \lambda_2\sigma_2(t), \tag{2.4}$$

for all times t, constants λ_1, λ_2 and strain histories $\varepsilon_1(s)$, $\varepsilon_2(s)$, $s \in (-\infty, t]$, the response is said to be *linear*. In other terms, the assumption of linearity of response states that if a strain history $\varepsilon(s)$ is scaled by constant λ, then the corresponding stress $\sigma(t)$ is also scaled by λ and if two strain histories are superposed, then the corresponding stresses are also superposed. A useful corollary of linearity of response is that there is no interaction between the stress history responses to separate strain histories. This assumption is assumed to be reasonable when the magnitude of the strain has been small for all past times, i.e. $|\varepsilon(s)| << 1$ for $s \in (-\infty, t]$.

It is important to note that the property of linearity of response does not refer to the shape of any material response curve. It refers to a method of constructing the stress response to a composite strain history by scaling and superposing the stress responses to the component strain histories. Scaling and superposition have convenient graphical interpretations that lead to important and useful tests for determining if the mechanical response of a material can be regarded as being linear. For a discussion of these, see [6, pp. 17–24].

2.6. Consequences of Linearity

In the remainder of this article, it is assumed that the material does not age. A discussion of linear aging viscoelastic materials parallel to that presented below is given in [7]. When the material is non-aging, it can be shown that the parameter t does not appear explicitly in the response functionals $\hat{\mathcal{G}}$ and $\hat{\mathcal{J}}$. In addition, Equations (2.1) and (2.2) become, respectively,

$$\sigma(t) = \mathcal{G}\left[\varepsilon(t-s)|_{s=0}^{\infty}\right], \tag{2.5_1}$$

$$\varepsilon(t) = \mathcal{J}\left[\sigma(t-s)|_{s=0}^{\infty}\right]. \tag{2.5_2}$$

In the histories appearing in the arguments of ($2.5_{1,2}$), the time variable s is measured backwards from the current time *t*. Its physical dimension can be thought of being "seconds ago".

Let $H(t)$ denote the Heaviside step function, i.e. $H(t) = 0$, $t \in (-\infty, 0)$ and $H(t) = 1$, $t \in [0, \infty)$. When there is linearity, the stress response to the step strain history $\varepsilon(s) = \varepsilon_o H(t)$ is

$$\sigma(t) = G(t, \varepsilon_o) = \mathcal{G}\left[\varepsilon_o H(t-s)|_{s=0}^{\infty}\right] = \varepsilon_o \mathcal{G}\left[H(t-s)|_{s=0}^{\infty}\right], \tag{2.6}$$

where the last step follows from the scaling property. Define

$$G(t) = \mathcal{G}\left[H(t-s)|_{s=0}^{\infty}\right]. \tag{2.7}$$

Equations (2.6) and (2.7) show that when there is linearity of response, $G(t, \varepsilon_o)$ is linear in ε_o,

$$G(t, \varepsilon_o) = \varepsilon_o G(t). \tag{2.8}$$

$G(t)$ is called the *stress relaxation modulus*. It is assumed that $G(0) > 0$ and $G(t)$ monotonically decreases to a non-zero positive limit as $t \to \infty$. It is convenient to introduce the notation $G(0) = G_o$ and G_∞ for the limit of $G(t)$ as $t \to \infty$. Gurtin and Sternberg [8], in their fundamental article, show that scaling and superposition lead to the representation for (2.5_1) as a Stieltjes integral,

$$\sigma(t) = \int_{-\infty}^{t} G(t-s)\mathrm{d}\varepsilon(s). \tag{2.9}$$

In a similar manner, the strain response to the step stress history $\sigma(s) = \sigma_o 1(t)$ is

$$\varepsilon(t) = J(t, \sigma_o) = \mathcal{J}\left[\sigma_o H(t-s)|_{s=0}^{\infty}\right] = \sigma_o \mathcal{J}\left[H(t-s)|_{s=0}^{\infty}\right]. \tag{2.10}$$

Denote

$$J(t) = \mathcal{J}\left[H(t-s)|_{s=0}^{\infty}\right]. \tag{2.11}$$

Then, the creep function $J(t, \sigma_o)$ is linear in σ_o,

$$J(t, \sigma_o) = \sigma_o J(t). \tag{2.12}$$

$J(t)$ is called the *creep compliance*. It is assumed that $J(0) > 0$ and $J(t)$ monotonically increases to a finite limit $J_\infty > 0$ as $t \to \infty$. It is convenient to introduce the notation $J(0) = J_o$ and J_∞ for the limit of $J(t)$ as $t \to \infty$. It is shown in [8] that scaling and superposition lead to the following representation for (2.5_2):

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-s)\mathrm{d}\sigma(s). \tag{2.13}$$

The constitutive equations (2.9) and (2.13) are written in the form of Stieltjes convolutions [8] in order to account for jump discontinuities in their arguments. When there is a jump in the stress or strain histories at $t = 0$ and the histories are differentiable for $t > 0$, (2.9) and (2.13) reduce to

$$\sigma(t) = \varepsilon(0)G(t) + \int_0^t G(t-s)\frac{\mathrm{d}\varepsilon(s)}{\mathrm{d}s}\mathrm{d}s, \tag{2.14$_1$}$$

$$\varepsilon(t) = \sigma(0)J(t) + \int_0^t J(t-s)\frac{\mathrm{d}\sigma(s)}{\mathrm{d}s}\mathrm{d}s. \tag{2.14$_2$}$$

Alternate forms for ($2.14_{1,2}$) can be obtained by integration by parts and a change of the integration variable,

$$\sigma(t) = \varepsilon(t)G(0) + \int_0^t \frac{\mathrm{d}G(s)}{\mathrm{d}s}\varepsilon(t-s)\mathrm{d}s = \varepsilon(t)G(0) + \int_0^t \frac{\mathrm{d}G(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)\mathrm{d}s, \tag{2.15$_1$}$$

$$\varepsilon(t) = \sigma(t)J(0) + \int_0^t \frac{\mathrm{d}J(s)}{\mathrm{d}s}\sigma(t-s)\mathrm{d}s = \sigma(t)J(0) + \int_0^t \frac{\mathrm{d}J(t-s)}{\mathrm{d}(t-s)}\sigma(s)\mathrm{d}s. \tag{2.15$_2$}$$

A complete list of alternate forms is given in [6, pp. 64–65].

As a concluding comment for this section, note that (2.8) and (2.12) show that the stress relaxation isochrone at time t is a straight line with slope $G(t)$ and the creep isochrone at time t is a straight line with slope $1/J(t)$.

2.7. Mechanical Analogs

Another approach used to develop constitutive equations for linear viscoelastic response involves mechanical analogs. These are mechanical devices formed by combining linear elastic springs and linear viscous dampers in series or parallel. The devices can be shown to exhibit a time dependent response that is similar to that observed in viscoelastic materials, namely, creep under constant load and force relaxation under constant deformation. For this reason these devices are treated as mechanical analogs of viscoelastic response. Since the springs and dampers are described by linear equations, as are the equations for the kinematics of deformation and force transmission, there is a linear relation between the overall force and deformation. These are interpreted as relations between stress and strain for a material and have the form

$$p_n\frac{\mathrm{d}^n\sigma}{\mathrm{d}t^n} + p_{n-1}\frac{\mathrm{d}^{n-1}\sigma}{\mathrm{d}t^{n-1}} + \cdots + p_1\frac{\mathrm{d}\sigma}{\mathrm{d}t} + p_o\sigma = q_n\frac{\mathrm{d}^n\varepsilon}{\mathrm{d}t^n} + q_{n-1}\frac{\mathrm{d}^{n-1}\varepsilon}{\mathrm{d}t^{n-1}} + \cdots + q_1\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + q_o\varepsilon, \tag{2.16}$$

where $p_o, q_o, p_1, q_1, \ldots, p_n, q_n$ are constants to be determined by experiments.

Equation (2.16) is valid only when the strain or stress histories are sufficiently smooth. When either has a jump discontinuity, as might occur at $t = 0$, (2.16) must be supplemented by appropriate jump relations. The appropriate relations between the initial conditions on the stress and strain and their first n-1 derivatives, that are consistent with (2.16), were developed in [8]. A complete statement of the constitutive equation obtained from the use of mechanical analogs then consists of both an equation of the form (2.16) and a set of appropriate initial conditions. This point is often overlooked in applications. Any constitutive equation of form (2.16), along with the appropriate initial conditions, can be expressed in either the form (2.14_1) or (2.14_2). On the other hand, a constitutive equation of form (2.14_1) or (2.14_2) can be reduced to the form (2.16) if and only if the creep compliance $J(t)$ and the stress relaxation modulus $G(t)$ satisfy specific conditions. A detailed discussion of this point can be found in [8]. Constitutive equation (2.16) gives equal emphasis to the stress and the strain. It therefore can be used to obtain the dual constitutive equations and their inverses.

2.8. Relation Between $G(t)$ and $J(t)$

When the step stress history, $\sigma(t) = \sigma_o$, $t > 0$, and the corresponding creep strain $\varepsilon(t) = \sigma_o J(t)$ given by (2.12) are substituted into (2.9) or (2.14_1), the result is

$$1 = \int_{-\infty}^{t} G(t-s)\mathrm{d}J(s), \tag{2.17_1}$$

or

$$1 = J(0)G(t) + \int_{0}^{t} G(t-s)\frac{\mathrm{d}J(s)}{\mathrm{d}s}\mathrm{d}s \tag{2.17_2}$$

Similarly, when the step strain history, $\varepsilon(t) = \varepsilon_o$, $t > 0$, and the corresponding stress relaxation response $\sigma(t) = \varepsilon_o G(t)$ given by (2.12) are substituted into (2.13) or (2.14_2), the result is

$$1 = \int_{-\infty}^{t} J(t-s)\mathrm{d}G(s) \tag{2.18_1}$$

or

$$1 = G(0)J(t) + \int_{0}^{t} J(t-s)\frac{\mathrm{d}G(s)}{\mathrm{d}s}\mathrm{d}s. \tag{2.18_2}$$

Equations $(2.17_{1,2})$ and $(2.18_{1,2})$ can be transformed into each other by an integration by parts. They establish alternate forms of a relation between $G(t)$ and $J(t)$. If $J(t)$ is known, then (2.17_2) is a linear Volterra integral equation for $G(t)$. Conversely, if $G(t)$ is known, (2.18_2) is a linear Volterra integral equation for $J(t)$. It is known that these equations have a unique solution. Because of this and the fact that (2.17_1) and (2.18_1) relate $G(t)$ and $J(t)$ by Stieltjes convolutions, Gurtin and Sternberg [8] refer to $G(t)$ and $J(t)$ as Stieltjes inverses of each other.

Thus, corresponding to a given stress relaxation modulus $G(t)$, there is a uniquely determined creep compliance $J(t)$, and vice versa. Several relations between their properties can be determined from (2.17_2) and (2.18_2), (see [6, pp. 67–71]),

$$G(t)J(t) \leq 1, \quad t \geq 0, \tag{2.19}$$

with

$$G_o J_o = 1, \quad G_\infty J_\infty = 1. \tag{2.20}$$

The simplest model of a linear viscoelastic solid that exhibits all of the important response characteristics, instantaneous elastic response, long-time or equilibrium elastic response and gradual stress relaxation, is the standard linear solid, also known as the three-parameter solid. Its stress relaxation modulus is given by

$$G(t) = G_\infty + [G_o - G_\infty]\,\mathrm{e}^{-t/\tau_R}, \tag{2.21}$$

where τ_R is called the characteristic stress relaxation time. The corresponding creep compliance, found by applying the Laplace transform to (2.18_2), is given by

$$J(t) = J_\infty + [J_o - J_\infty]\,\mathrm{e}^{-t/\tau_C}, \tag{2.22}$$

where τ_C is called the characteristic creep time. G_o, J_o, G_∞, J_∞ are related by (2.20) and the characteristic times are related by

$$\tau_C = \frac{G_o}{G_\infty}\tau_R. \tag{2.23}$$

Since stress relaxation implies $G_o/G_\infty > 1$, it follows from (2.23) that $\tau_C > \tau_R$.

Motivated by (2.21) and (2.22), the stress relaxation modulus $G(t)$ and creep compliance $J(t)$ are often decomposed into their long time equilibrium values and time dependent parts,

$$G(t) = G_\infty + \Delta G(t), \tag{2.24_1}$$

$$J(t) = J_\infty - \Delta J(t), \tag{2.24_2}$$

where $\Delta G(t) \to 0$ and $\Delta J(t) \to 0$ as $t \to \infty$.

Let Equations ($2.15_{1,2}$) be rewritten as

$$\sigma(t) = G(0)\varepsilon(t) + \int_0^t \frac{\mathrm{d}G(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)\mathrm{d}s, \tag{2.25_1}$$

$$\varepsilon(t) = J(0)\sigma(t) + \int_0^t \frac{\mathrm{d}J(t-s)}{\mathrm{d}(t-s)}\sigma(s)\mathrm{d}s. \tag{2.25_2}$$

Suppose that $G(t)$ and $J(t)$ are known. For a given stress history, (2.25_1) is a linear Volterra integral equation for the corresponding strain history. Conversely, for a given strain history, (2.25_2) is a linear Volterra integral equation for the corresponding stress history. It is straightforward to show, using (2.17_2) or (2.18_2) and elementary operations of calculus, that (2.25_2) is the solution to (2.25_1) and vice versa. Stated differently, (2.25_1) and ($2.2r_2$) are the inverses of each other. Thus, for non-aging, linear viscoelastic materials, the dual constitutive equations are also inverses.

2.9. Constant Strain or Stress Rate Histories

Consider the strain history given by $\varepsilon(t) = \alpha t$, $t \geq 0$, where α is a positive constant. When this is substituted into (2.14_1), the stress is given by

$$\sigma(t) = \alpha \int_0^t G(s)\mathrm{d}s. \tag{2.26}$$

Note that $\sigma(0) = 0$ and $\mathrm{d}\sigma(t)/\mathrm{d}t = \alpha G(t) > 0$. Since the stress relaxation modulus decreases monotonically with t, so does $\mathrm{d}\sigma(t)/\mathrm{d}t$. A plot of σ vs. t based on (2.26) thus increases from zero with a decreasing slope. It is common practice to express the time in terms of the strain, $t = \varepsilon/\alpha$, and substitute the result into (2.26), thereby relating σ to ε,

$$\sigma(\varepsilon) = \alpha \int_0^{\varepsilon/\alpha} G(s)\mathrm{d}s. \tag{2.27}$$

When there is stress relaxation, a plot of σ vs. ε based on (2.27) is not a straight line. The material is often described as having a nonlinear $\sigma - \varepsilon$ relation. In other words, a linear material appears to have a nonlinear stress-strain relation.

There are two comments to be made about this observation. First, when the adjective “linear” is used to describe material response, as in Section 2.4, it has a different meaning that when it used to describe the shape of the $\sigma - \varepsilon$ plot. A graphical construction in [6, p. 40] shows how a material that exhibits linearity of response and stress relaxation produces a plot of $\sigma - \varepsilon$ that is not a straight line. A $\sigma - \varepsilon$ plot that is a straight line occurs when there is no stress relaxation, i.e. $G(t) = G_o$, $t \geq 0$. In other words, it is misleading to use the shape of the $\sigma - \varepsilon$ plot to draw inferences about the material response. The second point is that each different strain history produces a different $\sigma - \varepsilon$ plot. This point can readily seen from (2.26), which shows that a new $\sigma - \varepsilon$ plot is obtained for each new value of the strain rate α.

Consider the stress history given by $\sigma(t) = \beta t$, $t \geq 0$, where β is a positive constant. From (2.14_2), the strain history is given by

$$\varepsilon(t) = \beta \int_0^t J(s)\mathrm{d}s. \tag{2.28}$$

Note that $\varepsilon(0) = 0$, $\mathrm{d}\varepsilon(t)/\mathrm{d}t = \beta J(t) > 0$ and it monotonically increases with t. The plot of ε vs. t is concave upwards. A plot of σ vs. ε is produced by expressing t in terms of σ, $t = \sigma/\beta$, and then substituting the result into (2.29),

$$\varepsilon = \beta \int_0^{\sigma/\beta} J(s)\mathrm{d}s. \tag{2.29}$$

The plot of σ vs. ε is not a straight line, but increases from the origin with a decreasing slope $1/J(\sigma/\beta)$. Different $\sigma - \varepsilon$ plots are produced by stress histories with different rates β. The constant strain rate $\sigma - \varepsilon$ plots do not generally coincide with the constant stress rate $\sigma - \varepsilon$ plot.

In summary, a plot of σ vs. ε for a non-aging, linear viscoelastic material provides limited information about its mechanical response.

2.10. Sinusoidal Strain Histories

A fundamental strain history used to study viscoelastic materials is the sinusoidal strain history,

$$\varepsilon(t) = \varepsilon_o \sin \omega t, \quad t \geq 0, \tag{2.30}$$

where ε_o is a constant such that $|\varepsilon_o| << 1$. The corresponding stress history is obtained by substituting (2.30) into (2.14_1). It can be shown that the stress reaches a state of steady sinusoidal oscillations described by

$$\sigma(t) = \varepsilon_o \left[G'(\omega) \sin \omega t + G''(\omega) \cos \omega t\right] \tag{2.31_1}$$

or

$$\sigma(t) = \varepsilon_o \left[G'(\omega)^2 + G''(\omega)^2\right]^{1/2} \sin(\omega t + \delta(\omega)), \tag{2.31_2}$$

where $\tan\delta(\omega) = G''(\omega)/G'(\omega)$. $G'(\omega)$ and $G''(\omega)$ are functions of frequency ω and are expressed in terms of the stress relaxation modulus by

$$G'(\omega) = G_\infty + \omega \int_0^\infty \Delta G(s) \sin \omega s \mathrm{d}s, \tag{2.32_1}$$

$$G''(\omega) = \omega \int_0^\infty \Delta G(s) \cos \omega s \mathrm{d}s, \tag{2.32_2}$$

where G_∞ and $\Delta G(t)$ were defined in (2.24_1).

It is seen from (2.31_1) or (2.31_2) that the stress varies sinusoidally with time at the same frequency ω as the strain, but with amplitude $\varepsilon_o \left[G'(\omega)^2 + G''(\omega)^2\right]^{1/2}$ and phase difference $\delta(\omega)$. $G'(\omega)$, the coefficient of the term in (2.31_1) in phase with the strain, is called the *storage modulus*. $G''(\omega)$, the coefficient of the term in (2.31_1) out of phase with the strain, is called the *loss modulus*. It can be shown using (2.32_2) that the phenomenon of stress relaxation implies that $G''(\omega) > 0$. It can also be shown that the work done on the material per cycle is $\varepsilon_o^2 \pi G''(\omega)$.

$G'(\omega)$ and $G''(\omega)$ are an alternate set of material properties and methods have been developed to measure them. They are defined by ($2.32_{1,2}$) in terms of the Fourier transform of $\Delta G(t)$. Consequently, $G(t)$ can be expressed in terms of $G'(\omega)$ and $G''(\omega)$ using the inverse Fourier transform,

$$G(t) = G_\infty + \frac{2}{\pi} \int_0^\infty \frac{G''(\omega)}{\omega} \cos \omega t \mathrm{d}\omega, \tag{2.33_1}$$

or

$$G(t) = \frac{2}{\pi} \int_0^\infty \frac{G'(\omega)}{\omega} \sin \omega t \mathrm{d}\omega. \tag{2.33_2}$$

It is common practice to use complex variables to describe the sinusoidal response of viscoelastic materials. Thus, instead of the strain history (2.30), one specifies $\varepsilon_o \exp(\mathrm{i}\omega t)$ where $\mathrm{i} = \sqrt{-1}$. The *complex modulus* $G^*(\omega)$ is defined by

$$G^*(\omega) = G'(\omega) + \mathrm{i}G''(\omega). \tag{2.34}$$

Then (2.31) is written

$$\sigma(t) = \varepsilon_o G^*(\omega)\mathrm{e}^{\mathrm{i}\omega t}. \tag{2.35}$$

Since the strain history in (2.30) is the imaginary part of $\varepsilon_o \exp(\mathrm{i}\omega t)$, the stress response is the imaginary part of (2.35).

It is also possible to specify the sinusoidal stress history

$$\sigma(t) = \sigma_o \sin \omega t, \tag{2.36}$$

the imaginary part of $\sigma_o \exp(\mathrm{i}\omega t)$. The strain history is

$$\varepsilon(t) = \sigma_o \left[J'(\omega) \sin \omega t + J''(\omega) \cos \omega t \right], \tag{2.37}$$

where $J'(\omega)$ and $J''(\omega)$ are components of the *complex compliance*

$$J^*(\omega) = J'(\omega) + \mathrm{i}J''(\omega). \tag{2.38}$$

The strain history in (2.37) is the imaginary part of $\sigma_o J^*(\omega)\mathrm{e}^{\mathrm{i}\omega t}$.

$J'(\omega)$ and $J''(\omega)$ can be expressed in terms of the creep compliance $J(t)$ by expressions that are analogous to $(2.32_{1,2})$. These are not presented here, but can be found in [6, p. 121]. It can be shown that $G^*(\omega)$ and $J^*(\omega)$ satisfy

$$G^*(\omega)J^*(\omega) = 1 \tag{2.39}$$

for all frequencies ω.

By use of (2.30) and (2.31_1), the stress σ can be expressed directly in terms of the strain ε by

$$\sigma^2 - 2G'\sigma\varepsilon + \left((G')^2 + (G'')^2 \right) \varepsilon^2 = \varepsilon_o^2 \, (G'')^2 . \tag{2.40}$$

This describes an ellipse whose properties depend on ω: (i) the enclosed area is $\varepsilon_o^2 \pi G''(\omega)$, (ii) the ε-axis intercept is $\varepsilon_o G'' / \left((G')^2 + (G'')^2 \right)^{1/2}$, (iii) the σ-axis intercept is $\sigma = \varepsilon_o G''$, and (iv) the maximum value of σ is $= \varepsilon_o \left[(G')^2 + (G'')^2 \right]^{1/2}$. The ellipse approaches the straight line $\sigma = G(0)\varepsilon$ as $\omega \to 0$ and the straight line $\sigma = G(\infty)\varepsilon$ as $\omega \to \infty$.

2.11. Constitutive Equation for Three Dimensional Response

Section 2 concludes with a statement of the constitutive equation for the three dimensional response of an isotropic non-aging linear viscoelastic material. The infinitesimal strain tensor is denoted by **e** and the stress tensor is denoted by **σ**.

It is shown in [8] that there are only two independent material properties, here chosen as a bulk modulus $K(t)$ and a shear modulus $\mu(t)$. The three-dimensional version of (2.14_1) is

$$\boldsymbol{\sigma}(t) = \mathbf{I}\left\{\left[K(t) - \frac{2}{3}\mu(t)\right]\mathrm{tr}\mathbf{e}(0) + \int_0^t \left[K(t-s) - \frac{2}{3}\mu(t-s)\right]\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{tr}\mathbf{e}(s)\mathrm{d}s\right\}$$
$$+\ 2\mu(t)\mathbf{e}(0) + 2\int_0^t \mu(t-s)\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{e}(s)\mathrm{d}s. \tag{2.41}$$

When $K(t)$ and $\mu(t)$ are independent of time, (2.41) reduces to the constitutive equation for a linear isotropic elastic material. It is also possible to state the three dimensional version of (2.14_2), that is, the dual to (2.41). This is not done here.

3. NONLINEAR VISCOELASTIC SOLIDS—KINEMATICS

This section presents the concepts for the kinematics of a body that underlie the study of viscoelasticity. See Spencer [9] or Atkin and Fox [10] for details.

A body is a set of material points called particles. A typical particle P is identified or labeled by its position vector **X** at some reference time t_o. The domain of **X** at time t_o is called a reference configuration of the body. It is assumed that a viscoelastic solid body has the same configuration for times $t < 0$, which is taken as its reference configuration. Let $\mathbf{x}(s)$ denote the position of particle P at a generic time $s \in (-\infty, t]$. The motion of particle P is described by the vector function

$$\mathbf{x}(s) = \mathbf{X}, \quad s \in (-\infty, 0)$$
$$\mathbf{x}(s) = \boldsymbol{\chi}(\mathbf{X}, s), \quad s \in [0, t]. \tag{3.1}$$

For a fixed **X**, (3.1) gives the path of particle P as time s increases. At a fixed time s, (3.1) gives the positions $\mathbf{x}(s)$ of all particles of the body. The domain of $\mathbf{x}(s)$ at time s is called the configuration of the body at time s. This motion is assumed to be invertible so that the label **X** of a particle can be expressed in terms of its position $\mathbf{x}(s)$ at time s,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}(s), s), \quad s \in [0, t]. \tag{3.2}$$

Let (3.2) be evaluated at time t and then substituted into (3.1). This introduces a description of the motion relative to the current configuration

$$\mathbf{x}(s) = \boldsymbol{\chi}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}(t), t), s\right) = \hat{\boldsymbol{\chi}}(\mathbf{x}(t), t, s). \tag{3.3}$$

The velocity and acceleration of particle P at time t are given by

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t}, \tag{3.4}$$

$$\ddot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}, \tag{3.5}$$

where the superposed dot denotes the partial derivative with respect to time holding the particle label fixed. When the independent spatial variable is $\mathbf{X}$, (3.4) and (3.5) give the material (or Lagrangian) description of the velocity and acceleration. Relation (3.2) evaluated at time t can be used to change the independent spatial variable in (3.4) and (3.5) from $\mathbf{X}$ to $\mathbf{x}(t)$ giving

$$\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right), \tag{3.6}$$

$$\mathbf{a}(\mathbf{x}, t) = \ddot{\mathbf{x}}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t\right). \tag{3.7}$$

Relations (3.6) and (3.7) give the spatial (or Eulerian) description of the velocity and acceleration.

By (3.1), the deformation gradient history is

$$\mathbf{F}(s) = \mathbf{I}, \quad s \in (-\infty, 0)$$
$$\mathbf{F}(s) = \frac{\partial \mathbf{x}(s)}{\partial \mathbf{X}}, \quad s \in [0, t]. \tag{3.8}$$

The velocity gradient at time t

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}. \tag{3.9}$$

is related to the deformation gradient by use of (3.6) and (3.8),

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{3.10}$$

$\mathbf{F}(s)$ contains information that compares the rotation and distortion in the neighborhood of a material particle at time s to its neighborhood in the reference configuration. It is assumed that

$$\det \mathbf{F}(s) > 0, \quad s \in [0, t]. \tag{3.11}$$

$\det \mathbf{F}(s)$ represents the ratio of the volume of the neighborhood of a particle at time s to that in the reference configuration.

Application of the Polar Decomposition Theorem of linear algebra leads to

$$\mathbf{F}(s) = \mathbf{R}(s)\mathbf{U}(s) = \mathbf{V}(s)\mathbf{R}(s), \quad s \in [0, t], \tag{3.12}$$

where the factors $\mathbf{U}(s)$, $\mathbf{V}(s)$, and $\mathbf{R}(s)$ satisfy

$$\mathbf{R}(s)\mathbf{R}(s)^T = \mathbf{R}(s)^T\mathbf{R}(s) = \mathbf{I}, \tag{3.13}$$

$$\mathbf{U}(s) = \mathbf{U}(s)^T, \quad \mathbf{V}(s) = \mathbf{V}(s)^T. \tag{3.14}$$

The orthogonal tensor $\mathbf{R}(s)$ represents the rigid body rotation of the neighborhood of the particle while $\mathbf{U}(s)$ and $\mathbf{V}(s)$, called the right and left stretch tensors, describe the local deformation of the neighborhood. It is tedious to compute tensors $\mathbf{R}(s)$ and $\mathbf{U}(s)$ from $\mathbf{F}(s)$. For this reason, one introduces the more easily computed tensor,

$$\mathbf{C}(s) = \mathbf{F}(s)^T\mathbf{F}(s) = \mathbf{U}(s)^2. \tag{3.15}$$

The tensor $\mathbf{C}(s)$, called the *right Cauchy–Green strain tensor*, has the same principal directions as $\mathbf{U}(s)$ and its principal values are the squares of those of $\mathbf{U}(s)$. Hence, $\mathbf{C}(s)$ is regarded as containing the same information as $\mathbf{U}(s)$ about the local deformation of the neighborhood. Let a tensor $\mathbf{B}(t)$ be defined by

$$\mathbf{B}(t) = \mathbf{F}(t)\mathbf{F}(t)^T. \tag{3.16}$$

$\mathbf{B}(t)$ is called the *left Cauchy–Green tensor* and arises when considering isotropic materials.

In nonlinear viscoelasticity, $\mathbf{F}(s)$ is often decomposed as follows:

$$\mathbf{F}(s) = \mathbf{F}_t(s)\mathbf{F}(t), \quad 0 \leq s \leq t \tag{3.17}$$

where

$$\mathbf{F}_t(s) = \frac{\partial \mathbf{x}(s)}{\partial \mathbf{x}(t)}. \tag{3.18}$$

$\mathbf{F}_t(s)$ is called the *relative deformation gradient*, and is the deformation gradient associated with the description of the motion in (3.3). It follows from (3.11) that $\det \mathbf{F}_t(s) > 0$. $\mathbf{F}_t(s)$ contains information that gives the rotation and deformation of the neighborhood of a particle in the configuration at time s relative to its neighborhood in the configuration at time t. This information is obtained by applying the Polar Decomposition Theorem to $\mathbf{F}_t(s)$. Thus, as in (3.12)

$$\mathbf{F}_t(s) = \mathbf{R}_t(s)\mathbf{U}_t(s) = \mathbf{V}_t(s)\mathbf{R}_t(s), \quad s \in [0, t], \tag{3.19}$$

where $\mathbf{R}_t(s)$, $\mathbf{U}_t(s)$ and $\mathbf{V}_t(s)$ satisfy

$$\mathbf{R}_t(s)\mathbf{R}_t(s)^T = \mathbf{R}_t(s)^T\mathbf{R}_t(s) = \mathbf{I},$$

$$\mathbf{U}_t(s) = \mathbf{U}_t(s)^T, \quad \mathbf{V}_t(s) = \mathbf{V}_t(s)^T. \tag{3.20}$$

It is convenient to define the relative right Cauchy–Green strain tensor

$$\mathbf{C}_t(s) = \mathbf{F}_t(s)^T \mathbf{F}_t(s). \tag{3.21}$$

4. FIELD EQUATIONS

Nonlinear viscoelastic solids, being materials whose mechanical response depends on the history of the motion, satisfy the field equations and boundary conditions of continuum mechanics at each time t in the configuration and surface occupied by the body at that time. The field equations are stated here in local form. The arguments of field variables are not stated for ease of presentation. For further details, see Spencer [9] or Ogden [11].

4.1. Conservation of Mass

Let ρ_o and ρ denote the mass per unit volume at a particle in the reference and current configurations, respectively. Conservation of mass requires that

$$\rho \det \mathbf{F} = \rho_o. \tag{4.1}$$

4.2. Conservation of Linear and Angular Momentum

The body force per unit mass on a material particle in the current configuration is denoted by **b**, the unit outer normal to an area element on the surface of the current configuration is denoted by **n**, the surface traction or force per unit area on this surface area element is denoted by **T** and the Cauchy or true stress tensor is denoted by $\boldsymbol{\sigma}$. Application of the Principles of the Conservation of Linear Momentum leads to

$$\mathbf{T} = \boldsymbol{\sigma}^T \mathbf{n} \tag{4.2}$$

on the current surface and

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a} \tag{4.3}$$

at each point within the current configuration. The Principle of the Conservation of Angular Momentum leads to the statement that the Cauchy stress tensor is symmetric,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{4.4}$$

4.3. Conservation of Energy

The internal energy per unit mass at a material particle is denoted by “e”, the rate of heat supply per unit mass to a particle is denoted by “r”, the heat energy per unit time (heat flux) per unit area through a surface area element of the current configuration with unit outer normal **n** is denoted by "q", and the heat flux vector is denoted by **q**. Application of the Principle of the Conservation of Energy in the current configuration leads to

$$q = \mathbf{q}^T \mathbf{n} \tag{4.5}$$

on the current surface and to

$$\rho \dot{e} = \mathrm{tr}\boldsymbol{\sigma}\dot{\mathbf{F}}\mathbf{F}^{-1} + \rho r - \mathrm{div}\mathbf{q} \tag{4.6}$$

at each point within the current configuration.

4.4. Entropy Inequality

The absolute temperature is denoted by θ and the entropy per unit mass of a particle is denoted by η. It is assumed that the entropy satisfies the Clausius–Duhem inequality, whose local form is

$$\rho \dot{\eta} \geq \frac{\rho r}{\theta} - \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right). \tag{4.7}$$

The current configuration is usually unknown and is determined as part of the process of solving a particular problem. Consequently, the field equations and boundary conditions are often transformed so that they are stated in the known reference configuration. Their statement is omitted here. For a detailed derivation of the equations, see Ogden [11].

5. CONSTITUTIVE THEORY FOR NONLINEAR VISCOELASTIC SOLIDS

The constitutive theory for nonlinear viscoelastic materials is summarized in this section. A thorough treatment can be found in Noll [12] and the fundamental treatise by Truesdell and Noll [13].

For a viscoelastic solid the constitutive assumption states that the stress $\boldsymbol{\sigma}$, internal energy "e" and specific entropy η at time t depend on histories of the deformation gradient **F**, temperature 'θ' and temperature gradient. Thermodynamic arguments show that the stress, internal energy, and specific entropy do not depend on the temperature gradient gradθ. This development is not presented here. Instead, the emphasis is on presenting the tensorial structure of the constitutive equations. Temperature, being a scalar, plays no role in determining this tensorial structure and will not be explicitly mentioned.

As in the case of linear viscoelasticity, it is assumed that the solid is in its reference configuration for $t < 0$, i.e. $\mathbf{F}(t) = \mathbf{I}$, $t < 0$. It is further assumed that the material does not age and the stress at the current time t depends on the history of the deformation gradient, that is, on all values of $\mathbf{F}(s)$, $s \in (-\infty, t]$, thereby allowing for jump discontinuities at $t = 0$. This constitutive equation expressing this dependence is denoted by

$$\boldsymbol{\sigma}(t) = \mathcal{F}\left[\mathbf{F}(t-s)|_{s=0}^{\infty}\right], \tag{5.1}$$

a generalization of (2.5). $\mathcal{F}$ is called a *tensor-valued response functional*. There are three main sources of restrictions on $\mathcal{F}$: (a) the influence of superposed rigid body motions, (b) material symmetry, (c) restrictions due to thermodynamics. For present purposes, the only restrictions considered here are those due to the influence of superposed rigid body motions and material symmetry.

5.1. Influence of superposed rigid body motions

Consider the motion $\mathbf{x}(s) = \boldsymbol{\chi}(\mathbf{X}, s)$, $s \in [0, t]$ in (3.1). Suppose that the body undergoes a second motion $\mathbf{x}(s) = \boldsymbol{\chi}^*(\mathbf{X}, s)$ that is obtained from the first by a superposed rigid body motion,

$$\boldsymbol{\chi}^*(\mathbf{X}, s) = \mathbf{Q}(s)\left[\boldsymbol{\chi}(\mathbf{X}, s) - \mathbf{d}(s)\right], \quad s \in [0, t]. \tag{5.2}$$

Vector $\mathbf{d}(s)$ represents a rigid body translation. $\mathbf{Q}(s)$ represents a rigid body rotation and satisfies

$$\mathbf{Q}(s)\mathbf{Q}(s)^T = \mathbf{Q}(s)^T\mathbf{Q}(s) = \mathbf{I}. \tag{5.3}$$

It is assumed that the superposed rigid body motion affects the stress at time t by only its rotation at time t. This leads to the condition that

$$\mathcal{F}\left[\mathbf{Q}(t-s)\mathbf{F}(t-s)|_{s=0}^{\infty}\right] = \mathbf{Q}(t)\mathcal{F}\left[\mathbf{F}(t-s)|_{s=0}^{\infty}\right]\mathbf{Q}(t)^T \tag{5.4}$$

for any rotation history $\mathbf{Q}(s)$ as long as it satisfies (5.3). This, when combined with the Polar Decomposition of $\mathbf{F}(s)$ in (3.12) leads to the statement that (5.1) is of the form

$$\boldsymbol{\sigma}(t) = \mathbf{R}(t)\mathcal{F}\left[\mathbf{U}(t-s)|_{s=0}^{\infty}\right]\mathbf{R}(t)^T. \tag{5.5}$$

Because (i) the determination of $\mathbf{R}(s)$ and $\mathbf{U}(s)$ from $\mathbf{F}(s)$ using (3.12) is tedious, (ii) $\mathbf{U}(s)$ and $\mathbf{C}(s)$ contain essentially the same information about the local deformation and (iii) $\mathcal{F}$ is as yet arbitrary, (5.5) is usually restated without loss in generality in the form

$$\boldsymbol{\sigma}(t) = \mathbf{F}(t)\mathcal{G}\left[\mathbf{C}(t-s)|_{s=0}^{\infty}\right]\mathbf{F}(t)^T, \tag{5.6}$$

where $\mathcal{G}$ is a new response functional. It is straightforward to show that (5.6) satisfies (5.4).

5.2. Material symmetry

The concept of material symmetry arises from the fact that a material has some physical microstructure in its reference configuration, such as a crystalline structure or a randomly oriented macromolecular network. Consider a sample of material in its reference configuration and its microstructure. Suppose there is a transformation of this reference configuration to a new configuration such that the material appears to have the same microstructure as before. Let both the original and transformed configurations be subjected to the same homogeneous deformation history with deformation gradient $\mathbf{F}(s)$. The underlying microstructures, which appear to be the same in their respective reference configurations, are distorted in the same way. The stresses are assumed to be the same at each time t and these configurations are said to be mechanically equivalent.

A transformation of the original reference configuration to one that is mechanically equivalent is a linear transformation denoted by $\mathbf{H}$. One restriction on $\mathbf{H}$ is that it produce no volume change and this leads to the condition that $|\det \mathbf{H}| = 1$. In addition, for most equivalent microstructures of interest, $\mathbf{H}$ is a rotation or a reflection and satisfies

$$\mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H} = \mathbf{I}. \tag{5.7}$$

Symmetries of a material are described by specifying the set of transformations $\mathbf{H}$ that lead to equivalent microstructures. These form a mathematical entity called a *material symmetry group*. The material symmetries commonly used to describe nonlinear viscoelastic materials are isotropy, transverse isotropy and orthotropy.

ISOTROPY: An isotropic material exhibits the same response with respect to all directions associated with its microstructure. A material is said to have hemihedral or proper isotropy if the transformations $\mathbf{H}$ of the material symmetry group are rotations. It is said to have holohedral or full isotropy if the material symmetry group consists of rotations and a central reflection.

Materials with anisotropy have specific directions associated with their microstructure. Let $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ be an orthonormal set of vectors that define these directions in the reference configuration.

TRANSVERSE ISOTROPY: A material is said to have transverse isotropy with respect to the direction indicated by, say, $\mathbf{E}_3$ if the material symmetry group consists of rotations about $\mathbf{E}_3$. Different classes of transverse isotropy arise by also including reflections about planes perpendicular to these vectors.

ORTHOTROPY: A material is said to have orthotropy if the material symmetry group contains 90° rotations about each of the vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. Different classes of orthotropy arise by also including reflections about planes perpendicular to these vectors.

For each transformation $\mathbf{H}$ of a material symmetry group, the above discussion implies that the constitutive equation (5.1) must satisfy restriction

$$\mathcal{F}\left[\mathbf{F}(t-s)|_{s=0}^{\infty}\right] = \mathcal{F}\left[\mathbf{F}(t-s)\mathbf{H}|_{s=0}^{\infty}\right]. \tag{5.8}$$

Material symmetry restrictions can be imposed on the response functional $\mathcal{G}$ by substituting (5.6) into (5.8) giving

$$\mathbf{H}^T\mathcal{G}\left[\mathbf{C}(t-s)|_{s=0}^{\infty}\right]\mathbf{H} = \mathcal{G}\left[\mathbf{H}^T\mathbf{C}(t-s)\big|_{s=0}^{\infty}\mathbf{H}\right]. \tag{5.9}$$

Note that this restriction is imposed by transformations of the reference configuration and is independent of the subsequent deformation history. In other terms, the proper statement of a material symmetry is that a material is isotropic, transversely isotropic or orthotropic with respect to its reference configuration. The statement is sometimes made that a deformation causes a material to become anisotropic. A more precise statement is that there is a material symmetry group associated with the current configuration that is determined using Noll's rule [12] from the symmetry group associated with the reference configuration and the current deformation gradient.

5.3. Constraints

The possible motions of a body may be limited by constraints such as incompressibility or inextensibility in certain directions. Such constraints impose restrictions on the constitutive equations. Discussion here is restricted to the constraint of incompressibility.

In many polymeric materials, the volume change during deformation is observed to be very small. By (3.11) and (4.1), this leads to an idealized material model for which any possible motion must satisfy the constraint,

$$\det \mathbf{F}(s) = 1, \quad s \in [0, t]\,. \tag{5.10}$$

Motions that satisfy (5.10) are described as being *isochoric*. Consideration of the restrictions of the thermodynamics for materials with constraint (5.10) leads to a modified form for constitutive equation (5.1),

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \mathcal{F}\left[\mathbf{F}(t-s)|_{s=0}^{\infty}\right], \tag{5.11}$$

in which p is an arbitrary scalar. The restriction imposed by consideration of the influence of superposed rigid body motions must still be satisfied so that $\mathcal{F}$ in (5.11) still must satisfy (5.4). Equations (5.5) and (5.6) then become

$$\boldsymbol{\sigma}(t) \;=\; -p\mathbf{I} + \mathbf{R}(t)\mathcal{F}\left[\mathbf{U}(t-s)|_{s=0}^{\infty}\right]\mathbf{R}(t)^{T}, \tag{5.12$_1$}$$

$$\boldsymbol{\sigma}(t) \;=\; -p\mathbf{I} + \mathbf{F}(t)\mathcal{G}\left[\mathbf{C}(t-s)|_{s=0}^{\infty}\right]\mathbf{F}(t)^{T}. \tag{5.12$_2$}$$

in which $\det \mathbf{U}(s) = \det \mathbf{C}(s) = 1$. Similarly, material symmetry considerations imply that the response functionals $\mathcal{F}$ and $\mathcal{G}$ in ($5.12_{1,2}$) satisfy (5.8) and (5.9), respectively.

5.4. A Special Result for Isotropic Materials

There is an interesting result for isotropic nonlinear viscoelastic solids that does not depend on the form of the response functional $\mathcal{F}$. Since (5.4) and (5.8) must hold for arbitrary rotation transformations $\mathbf{H}$ and rotation histories $\mathbf{Q}(s)$, $s \in [0, t]$, Noll [12] has shown that the constitutive equation can be written in the form

$$\boldsymbol{\sigma}(t) = \hat{\mathcal{G}}\left[\mathbf{B}(t); \mathbf{C}_t(t-s)|_{s=0}^{\infty}\right], \tag{5.13}$$

where $\mathbf{B}(t)$ was defined in (3.16) and $\mathbf{C}_t(s)$ was defined in (3.21). The response functional $\hat{\mathcal{G}}$ satisfies

$$\hat{\mathcal{G}}\left[\mathbf{H}^T\mathbf{B}(t)\mathbf{H};\, \mathbf{H}^T\mathbf{C}_t(t-s)\mathbf{H}\big|_{s=0}^{\infty}\right] = \mathbf{H}^T\hat{\mathcal{G}}\left[\mathbf{B}(t);\, \mathbf{C}_t(t-s)|_{s=0}^{\infty}\right]\mathbf{H} \tag{5.14}$$

for all orthogonal transformations $\mathbf{H}$. $\hat{\mathcal{G}}$ is said to be an *isotropic functional*. If, in addition, the material is incompressible and isotropic, the constitutive equation can be written as

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \hat{\mathcal{G}}\left[\mathbf{B}(t);\, \mathbf{C}_t(t-s)|_{s=0}^{\infty}\right], \tag{5.15}$$

where $\hat{\mathcal{G}}$ satisfies (5.14).

6. SOME PROPOSED CONSTITUTIVE EQUATIONS FOR NONLINEAR VISCOELASTIC SOLIDS

There is no generally accepted well-defined form for the constitutive equations (5.5) and (5.6) for nonlinear viscoelastic solids as there is for linear viscoelastic solids. A number of specific representations for the response functionals $\mathcal{F}$ and $\mathcal{G}$ have appeared in the literature and these are summarized in the book by Lockett [2] and the recent review article by Drapaca et al. [5]. The latter also summarizes the mathematical issues used in the development of the representations. Attention is restricted, in this section, to a presentation of the mathematical forms of these representations, as restricted by consideration of superposed rigid body motions. The additional restrictions due to considerations of material symmetry are presented in later sections.

6.1. Rate and Differential Type Constitutive Equations

One class of constitutive equations generalizes (2.16) to a relation between the stress and its first "n" time derivatives and the deformation gradient and its first "m" time derivatives, all evaluated at the current time t,

$$\mathcal{R}\left[\boldsymbol{\sigma}, \frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t}, \frac{\mathrm{d}^2\boldsymbol{\sigma}}{\mathrm{d}t^2}, \cdots \frac{\mathrm{d}^n\boldsymbol{\sigma}}{\mathrm{d}t^n}; \mathbf{F}, \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}, \frac{\mathrm{d}^2\mathbf{F}}{\mathrm{d}t^2}, \cdots, \frac{\mathrm{d}^m\mathbf{F}}{\mathrm{d}t^m}\right] = \mathbf{0}, \tag{6.1}$$

where $\mathcal{R}$ is a function of $m+n+2$ arguments. When subjected to the restrictions imposed by the considerations of superposed rigid body motions, the constitutive equation has the form [12]

$$\mathbf{R}_1\left[\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}, \mathbf{R}^T\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left[\mathbf{R}_t(s)^{\mathbf{T}}\boldsymbol{\sigma}(s)\mathbf{R}_t(s)\right]_{s=t}\mathbf{R}; \mathbf{U}, \mathbf{R}^T\left[\frac{\mathrm{d}^k}{\mathrm{d}s^k}\mathbf{U}_t(s)\right]_{s=t}\mathbf{R}\right] = \mathbf{0}, \tag{6.2}$$

in which $j = 1, \ldots, n$ and $k = 1, \ldots, m$. Such constitutive equations are said to be of *rate type* and

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left[\mathbf{R}_t(s)^{\mathbf{T}}\boldsymbol{\sigma}(s)\mathbf{R}_t(s)\right]_{s=t} \tag{6.3}$$

is called the *jth invariant stress rate.*

Equation (6.2) can be solved, in concept, for the stress in terms of the deformation history or for the deformation in terms of the stress history. It contains, in effect, both the dual constitutive equations and their inverses. As in the case of (2.16), (6.2) must be supplemented by a set of conditions relating the stress and kinematical variables at a jump discontinuity. These do not appear to have been developed.

A special case of (6.2) is explicit in the stress and does not depend on the stress rates,

$$\boldsymbol{\sigma} = \mathbf{R}\tilde{\mathbf{W}}\left[\mathbf{U}, \mathbf{R}^T\left[\frac{\mathrm{d}^j}{\mathrm{d}s^j}\mathbf{U}_t(s)\right]_{s=t}\mathbf{R}\right]\mathbf{R}^T. \tag{6.4}$$

This constitutive equation is said to be of *differential type.* It is assumed to be useful when the stress depends on $\mathbf{F}(s)$ for values of s near the current time t, i.e. on the recent past. $\mathbf{F}(s)$ can then be approximated by the first n terms of its power series.

6.2. Green–Rivlin Multiple Integral Constitutive Equations

Consider the Green–St. Venant strain tensor defined by

$$\mathbf{E}(s) = \frac{1}{2}\left(\mathbf{C}(s) - \mathbf{I}\right). \tag{6.5}$$

Note by (3.8) that $\mathbf{E}(s) = \mathbf{0}$, $s \in (-\infty, 0)$. Let $\mathbf{E}(s)$ be introduced into (5.6), which then becomes

$$\boldsymbol{\sigma}(t) = \mathbf{F}(t)\mathcal{G}_1\left[\mathbf{E}(t-s)|_{s=0}^{\infty}\right]\mathbf{F}(t)^T. \tag{6.6}$$

Green and Rivlin [14] assumed that the response functional $\mathcal{G}_1$ is continuous in $\mathbf{E}(s)$ in a sense described in [5]. By expressing $\mathbf{E}(s)$, $s \in [0, t]$ as a Fourier series and then using the Stone–Weierstrass theorem, Green and Rivlin obtained a representation for (6.6) as a multiple integral series,

$$\begin{aligned}\mathcal{G}_1\left[\mathbf{E}(t-s)|_{s=0}^{\infty}\right] &= \int_{-\infty}^{t}\mathbf{K}_1(t-s_1)\mathrm{d}\mathbf{E}(s_1)\\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t}\mathbf{K}_2(t-s_1, t-s_2)\mathrm{d}\mathbf{E}(s_1)\mathrm{d}\mathbf{E}(s_2)\\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t}\mathbf{K}_3(t-s_1, t-s_2, t-s_3)\mathrm{d}\mathbf{E}(s_1)\mathrm{d}\mathbf{E}(s_2)\mathrm{d}\mathbf{E}(s_3) + \cdots.\end{aligned} \tag{6.7}$$

This is written in the same form as in (2.9), i.e. in terms of Stieltjes convolutions, in order to account for a jump discontinuity in $\mathbf{E}(s)$. $\mathbf{K}_1(t-s_1)$, $\mathbf{K}_2(t-s_1, t-s_2)$, $\mathbf{K}_3(t-s_1, t-s_2, t-s_3)$ are tensor-valued functions of order four, six and eight, respectively.

The dual assumption can also be made. As discussed in Lockett [2], it has the form

$$\mathbf{E}(t) = \mathcal{J}_1 \left[\left(\mathbf{F}(s)^{-1} \boldsymbol{\sigma}(s) \mathbf{F}(s)^{-T} \right) \Big|_{0-}^{t} \right]. \tag{6.8}$$

$\mathcal{J}_1$ has a multiple integral series representation analogous to (6.7) with $\mathbf{E}(s)$ replaced by $\mathbf{F}(s)^{-1}\boldsymbol{\sigma}(s)\mathbf{F}(s)^{-T}$. In applications, only the truncation of (6.7) or the series representation of (6.8) up to triple integrals has been considered.

6.3. Finite Linear Viscoelasticity

Coleman and Noll [15] developed a constitutive equation based on the assumption of fading memory, i.e. the current stress depends more on recent deformations than past deformations. They also assumed that deformation of the current configuration with respect to the reference configuration is large, and that the deformation of recent configurations relative to the current configuration changes slowly, in a sense made precise in [15]. This led to Taylor series–like approximations to (5.5) and (5.6), the leading terms of which are

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathbf{R}(t) \Big\{ \mathbf{k}_1 \left[\mathbf{C}(t) \right] \\ &\quad + \int_{-\infty}^{t} \mathbf{K}_1 \left[\mathbf{C}(t), t-s \right] \left[\mathbf{R}(t)^T \left(\mathbf{C}_t(s) - \mathbf{I} \right) \mathbf{R}(t) \right] \mathrm{d}s \Big\} \mathbf{R}(t)^T \end{aligned} \tag{6.9$_1$}$$

and

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathbf{F}(t) \Big\{ \mathbf{k}_2 \left[\mathbf{C}(t) \right] \\ &\quad + \int_{-\infty}^{t} \mathbf{K}_2 \left[\mathbf{C}(t), t-s \right] \left[\mathbf{F}(t)^T \left(\mathbf{C}_t(s) - \mathbf{I} \right) \mathbf{F}(t) \right] \mathrm{d}s \Big\} \mathbf{F}(t)^T. \end{aligned} \tag{6.9$_2$}$$

The integrands in (6.9$_{1,2}$) are linear in the tensors $\mathbf{R}(t)^T\,(\mathbf{C}_t(s) - \mathbf{I})\,\mathbf{R}(t)$ and $\mathbf{F}(t)^T(\mathbf{C}_t(s) - \mathbf{I})\mathbf{F}(t)$. $\mathbf{K}_1$ and $\mathbf{K}_2$ are fourth-order tensor functions of "s" and $\mathbf{C}(t)$ and have the property, made precise in [15], that they monotonically decay to zero as s increases. Dependence on the finite strain tensor $\mathbf{C}(t)$ expresses the notion that deformation of the current configuration with respect to the reference configuration can be large. The linear dependence of the integrand on $\mathbf{C}_t(s) - \mathbf{I}$ arises from the assumption that the deformation occurs slowly.

If the material is assumed to be incompressible, then the assumption of fading memory imposed on (6.9$_{1,2}$) leads to

$$\begin{aligned}\boldsymbol{\sigma}(t) &= -p\mathbf{I} + \mathbf{R}(t) \Big\{ \mathbf{k}_1 \left[\mathbf{C}(t) \right] \\ &\quad + \int_{0}^{t} \mathbf{K}_1 \left[\mathbf{C}(t), t-s \right] \left[\mathbf{R}(t)^T \left(\mathbf{C}_t(s) - \mathbf{I} \right) \mathbf{R}(t) \right] \mathrm{d}s \Big\} \mathbf{R}(t)^T \end{aligned} \tag{6.10$_1$}$$

and

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \mathbf{F}(t)\left\{\mathbf{k}_2\left[\mathbf{C}(t)\right] + \int_0^t \mathbf{K}_2\left[\mathbf{C}(t), t-s\right]\left[\mathbf{F}(t)^T\left(\mathbf{C}_t(s) - \mathbf{I}\right)\mathbf{F}(t)\right]\mathrm{d}s\right\}\mathbf{F}(t)^T. \qquad (6.10_2)$$

When there is no deformation, then $\mathbf{x}(t) = \mathbf{X}$, $t > 0$. All of the tensorial variables appearing in $(6.9_{1,2})$ reduce to $\mathbf{I}$ and the stress reduces to $\boldsymbol{\sigma}(t) = \mathbf{k}_1(\mathbf{I})$ or $\boldsymbol{\sigma}(t) = \mathbf{k}_2(\mathbf{I})$. It is assumed that the material is stress free in its reference configuration so that $\mathbf{k}_1(\mathbf{I}) = \mathbf{k}_2(\mathbf{I}) = \mathbf{0}$.

The dual form of this constitutive equation in which the deformation is expressed in terms of the stress history has not been considered.

6.4. Pipkin–Rogers Constitutive Theory

Pipkin and Rogers [16] developed a constitutive theory for nonlinear viscoelastic solids based on a set of assumptions about the response to step strain histories. The response functional $\mathcal{G}$ in (5.6) has the form of a series in which the first term gives the best approximation to measured mechanical response using single step strain histories. The next level of approximation uses the response to double step strain histories, and so on. The leading term of the series is

$$\boldsymbol{\sigma}(t) = \mathbf{F}(t)\left\{\mathbf{K}_3\left[\mathbf{C}(t), 0\right] + \int_0^t \frac{\partial}{\partial (t-s)}\mathbf{K}_3\left[\mathbf{C}(s), t-s\right]\mathrm{d}s\right\}\mathbf{F}(t)^T. \qquad (6.11)$$

If the material is assumed to be incompressible, then

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \mathbf{F}(t)\left\{\mathbf{K}_3\left[\mathbf{C}(t), 0\right] + \int_0^t \frac{\partial}{\partial (t-s)}\mathbf{K}_3\left[\mathbf{C}(s), t-s\right]\mathrm{d}s\right\}\mathbf{F}(t)^T, \qquad (6.12)$$

where the motion must be such that $\det \mathbf{F}(t) = \det \mathbf{C}(s) = 1$.

At a fixed value $\mathbf{C}$ of the strain tensor argument, $\mathbf{K}_3\left[\mathbf{C}, s\right]$ is assumed to monotonically decrease with "s" to a non-zero limit. This, in effect, incorporates the notion of fading memory into the Pipkin–Rogers constitutive theory. If the material does not deform from its reference configuration, then (6.11) reduces to $\boldsymbol{\sigma}(t) = \mathbf{K}_3(\mathbf{I}, t)$. It is assumed that the material is stress free and hence $\mathbf{K}_3(\mathbf{I}, t) = 0$.

Pipkin and Rogers discussed the dual to (6.11). Although the dual formulation gives an expression that is convenient for modeling the results of creep experiments, it is less convenient in applications where (4.3) must be satisfied. Consequently, only (6.11) and (6.12) are considered here.

6.5. Quasi-Linear Viscoelasticity

The special case of (6.11) or (6.12) when $\mathbf{K}_3\,[\mathbf{C}, s]$ is separable, i.e.

$$\mathbf{K}_3\,[\mathbf{C}, s] = \mathbf{K}^{(e)}\,[\mathbf{C}]\,G(s) \tag{6.13}$$

has become known as *quasi-linear viscoelasticity*. $\mathbf{K}^{(e)}\,[\mathbf{C}]$ is normalized so that $G(0) = 1$. Then, (6.12) becomes

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \mathbf{F}(t)\left\{\mathbf{K}^{(e)}\,[\mathbf{C}(t)] + \int_0^t \mathbf{K}^{(e)}\,[\mathbf{C}(s)]\,\frac{\partial G(t-s)}{\partial\,(t-s)}\mathrm{d}s\right\}\mathbf{F}(t)^T. \tag{6.14}$$

The terminology "quasi-linear viscoelasticity" arises because $\mathbf{K}^{(e)}\,[\mathbf{C}]$ can be thought of as a nonlinear measure of strain. The expression in braces in (6.14) is linear in this nonlinear strain measure.

This constitutive equation, proposed by Fung [17], is used to represent the mechanical response of a variety of biological tissues. It is also convenient for developing analytical results that illustrate qualitative features of nonlinear viscoelastic behavior that could be expected when using more complicated constitutive equations.

7. MATERIAL SYMMETRY RESTRICTIONS ON THE PROPOSED CONSTITUTIVE EQUATIONS

The forms for the constitutive equations presented in Section 6 reduce the problem of finding material symmetry restrictions on the response functional $\mathcal{G}$ in (5.6) to that of finding material symmetry restrictions on the tensor valued functions in (6.2), (6.4), (6.7), ($6.9_{1,2}$), (6.10) or (6.11). Each of these is a tensor valued function of a set of tensors $\mathbf{M}_\mathrm{i}$, $\mathrm{i} = 1, 2, \cdots, N$, that is, of the form $\boldsymbol{\Phi}(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$. The material symmetry condition (5.9) imposed on the functions $\boldsymbol{\Phi}$ has the form

$$\mathbf{H}\boldsymbol{\Phi}(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)\mathbf{H}^T = \boldsymbol{\Phi}(\mathbf{H}\mathbf{M}_1\mathbf{H}^T, \mathbf{H}\mathbf{M}_2\mathbf{H}^T, \ldots, \mathbf{H}\mathbf{M}_N\mathbf{H}^T). \tag{7.1}$$

The method for determining the form of $\boldsymbol{\Phi}(\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_N)$ satisfying (7.1) has been presented in the review article by Spencer [18]. It is shown in [18] that for each type of material symmetry,

1. there is a set of basic scalar functions $I_k\,(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$, $k = 1, \ldots, K$, called *invariants*, that have the property

   $$I_k\,(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N) = I_k\left(\mathbf{H}\mathbf{M}_1\mathbf{H}^T, \mathbf{H}\mathbf{M}_2\mathbf{H}^T, \ldots, \mathbf{H}\mathbf{M}_N\mathbf{H}^T\right) \tag{7.2}$$

   for each transformation $\mathbf{H}$ of the material symmetry group under consideration,
2. there is a set of basic tensor valued functions, $\mathbf{P}_m(\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_N)$, $m = 1, \cdots, M$ that satisfy (7.1) for each transformation $\mathbf{H}$ of the material symmetry group under consideration.

3. A function $\mathbf{\Phi}(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$ that satisfies (7.1) can be represented in the form

$$\mathbf{\Phi} = \sum_{m=1}^{M} \tilde{\Phi}_m \mathbf{P}_m, \tag{7.3}$$

where $\tilde{\Phi}_m$ is a scalar function of the basic scalar invariants, i.e. $\tilde{\Phi}_m = \tilde{\Phi}_m\,(I_1, \ldots, I_N)$. It is straightforward to show that (7.3) satisfies (7.1).

The representation (7.3) shows that material symmetry restrictions determine the basic functions $\mathbf{P}_m(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$ and hence the general tensorial structure of $\mathbf{\Phi}(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$. Material symmetry restrictions also determine the arguments of the scalar coefficients $\tilde{\Phi}_m$, but provide no information as to how $\tilde{\Phi}_m$ depends on these arguments.

8. CONSTITUTIVE EQUATIONS FOR ISOTROPIC MATERIALS

The constitutive theories in Section 6 involve only second order tensors. In this case, there is no distinction between proper or full isotropy because (5.9) is identically satisfied by central reflection transformations. The following presents the forms of the constitutive equations in Section 6 when the material is isotropic.

8.1. Rate and Differential Type Constitutive Equations

The rate type constitutive equation in (6.2) becomes

$$\mathcal{R}_2\left[\boldsymbol{\sigma}, \overset{(1)}{\boldsymbol{\sigma}}, \overset{(2)}{\boldsymbol{\sigma}}, \cdots, \overset{(n)}{\boldsymbol{\sigma}}; \mathbf{A}_l, \mathbf{A}_2, \cdots, \mathbf{A}_m; \mathbf{B}\right] = \mathbf{0}. \tag{8.1}$$

The tensors $\mathbf{A}_k$, known as Rivlin–Ericksen tensors [19], are defined recursively by

$$\mathbf{A}_1 = (\mathbf{L} + \mathbf{L})^T, \quad \mathbf{A}_{k+1} = \frac{D\mathbf{A}_k}{Dt} + \mathbf{A}_k\mathbf{L} + \mathbf{L}^T\mathbf{A}_k, \tag{8.2}$$

where $\mathbf{L}$ was introduced in (3.9). $\overset{(n)}{\boldsymbol{\sigma}}$ is defined recursively by

$$\overset{(0)}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}, \qquad \overset{(n+1)}{\boldsymbol{\sigma}} = \frac{D\overset{(n)}{\boldsymbol{\sigma}}}{Dt} + \overset{(n)}{\boldsymbol{\sigma}}\mathbf{L} + \mathbf{L}^T\overset{(n)}{\boldsymbol{\sigma}}. \tag{8.3}$$

$\mathcal{R}_2$ is an isotropic function of its arguments whose general form can be constructed by identifying it with $\mathbf{\Phi}$ in Section 7. The general form is not presented here because rate type constitutive equations are rarely used in the description of viscoelastic solids, although they are used for viscoelastic fluids. They are not discussed further in this article.

The constitutive equation (6.4) for isotropic materials of differential type becomes

$$\boldsymbol{\sigma} = \mathcal{R}_3 \left[\mathbf{A}_l, \mathbf{A}_2, \cdots, \mathbf{A}_m; \mathbf{B}\right], \tag{8.4}$$

where $\mathcal{R}_3$ is an isotropic function of it arguments. Materials modeled by (8.4) are referred to as Rivlin–Ericksen materials [19]. A general form for $\mathcal{R}_3$ can be constructed by identifying it with $\boldsymbol{\Phi}$ in Section 7. This constitutive equation has been used to study limited aspects of the mechanics of viscoelastic solid. It will not receive further discussion.

8.2. Green–Rivlin Multiple Integral Constitutive Equations

The form of each integrand in (6.7) can be constructed by identifying it with $\boldsymbol{\Phi}$ in Section 7. For isotropic materials, the Green–Rivlin constitutive equation (6.7) becomes (see [2])

$$\begin{aligned} \mathbf{G}_1\left[\mathbf{E}(t-s)|_{s=0}^{\infty}\right] &= \int_{-\infty}^{t}\left[\psi_1 T_1 \mathbf{I} + \psi_2 \mathbf{M}_1\right] \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t}\left[\mathbf{I}\psi_3 T_1 T_2 + \mathbf{I}\psi_4 T_{12} + \psi_5 T_1 \mathbf{M}_2 + \psi_6 \mathbf{M}_1 \mathbf{M}_2\right] \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t}[\mathbf{I}\psi_7 T_{123} + \mathbf{I}\psi_8 T_1 T_{23} + \psi_9 T_1 T_2 \mathbf{M}_3 \\ &+ \psi_{10} T_{12} \mathbf{M}_3 + \psi_{11} T_1 \mathbf{M}_2 \mathbf{M}_3 + \psi_{12} \mathbf{M}_1 \mathbf{M}_2 \mathbf{M}_3] + \cdots \end{aligned} \tag{8.5}$$

where $\psi_{\rm i} = \psi_{\rm i}(t - s_1)$, i $= 1, 2$, $\psi_{\rm i} = \psi_{\rm i}(t - s_1, t - s_2)$, i $= 3, 4, 5, 6$, $\psi_{\rm i} = \psi_{\rm i}(t - s_1, t - s_2, t - s_3)$, i $= 7, \ldots, 12$, $\mathbf{M}_\alpha = \mathrm{d}\mathbf{E}(s_\alpha)$, $T_\alpha = \mathrm{tr}\,(\mathbf{M}_\alpha)$, $T_{\alpha\beta} = \mathrm{tr}\left(\mathbf{M}_\alpha \mathbf{M}_\beta\right)$ and $T_{\alpha\beta\gamma} = \mathrm{tr}\left(\mathbf{M}_\alpha \mathbf{M}_\beta \mathbf{M}_\gamma\right)$.

This constitutive theory received a great deal of attention when first proposed. An extensive discussion of experimental and analytical work based on this theory is provided in the book by Findley et al. [1]. Most of the experimental work makes use of the dual form (6.8) because it is experimentally more feasible to apply step stresses and measure creep. There is little current interest in the model for several reasons. The triple integral truncation of (8.5) is adequate for strains of about 0.1. However, larger strains require integrals of higher multiplicity. This rapidly increases the number of experiments and functions of time to be measured and the cost of the numerical evaluation of the integrals. This theory will not receive further discussion in later sections.

8.3. Finite Linear Viscoelasticity

When the material is isotropic, $(6.9_{1,2})$ can be put in the form of (5.13),

$$\boldsymbol{\sigma}(t) = \hat{\mathbf{k}}\left[\mathbf{B}(t)\right] + \int_{-\infty}^{t} \hat{\mathbf{K}}\left[\mathbf{B}(t), t-s\right]\left(\mathbf{C}_t(s) - \mathbf{I}\right)\mathrm{d}s, \tag{8.6}$$

in which the forms of $\hat{\mathbf{k}}$ and $\hat{\mathbf{K}}$ are found using the results of Section 7. $\hat{\mathbf{K}}[\mathbf{B}, s]$ has the property that it monotonically decreases with "s" to zero for fixed $\mathbf{B}$. In order to discuss the response to step changes in deformation, it is usually written in the alternate form obtained by integrating by parts,

$$\boldsymbol{\sigma}(t) = \tilde{\mathbf{k}}[\mathbf{B}(t)] + \int_{-\infty}^{t} \tilde{\mathbf{K}}[\mathbf{B}(t), t-s] \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} \mathrm{d}s. \tag{8.7}$$

The form of $\tilde{\mathbf{k}}$ is

$$\tilde{\mathbf{k}} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2. \tag{8.8}$$

The scalar coefficients α_{i} are functions of the invariants $I_\alpha(\mathbf{B})$ of $\mathbf{B}(t)$ defined by

$$I_1(\mathbf{B}) = \mathrm{tr}(\mathbf{B}), \quad I_2(\mathbf{B}) = \frac{1}{2}\left[\mathrm{tr}(\mathbf{B})^2 - \mathrm{tr}(\mathbf{B}^2)\right], \quad I_3(\mathbf{B}) = \det(\mathbf{B}). \tag{8.9}$$

The properties of $\tilde{\mathbf{K}}[\mathbf{B}, s]$ are similar to those of $\hat{\mathbf{K}}[\mathbf{B}, s]$. The integrand of (8.7) is given

$$\begin{aligned}\tilde{\mathbf{K}}[\mathbf{B}(t), t-s] \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} &= \sum_{\alpha=0}^{2} \phi_\alpha(t-s) \left[\mathbf{B}^\alpha \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} + \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} \mathbf{B}^\alpha\right] \\ &+ \sum_{\alpha=0}^{2} \sum_{\beta=0}^{2} \phi_{\alpha\beta}(t-s) \mathbf{B}^\alpha \mathrm{tr}\left[\mathbf{B}^\beta \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s}\right].\end{aligned} \tag{8.10}$$

The scalar coefficients ϕ_α and $\phi_{\alpha\beta}$ are functions of $t-s$ and the invariants $I_\alpha(\mathbf{B})$.

If the material is assumed to be incompressible and isotropic, (8.6) and (8.7) are replaced, respectively, by

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \hat{\mathbf{k}}[\mathbf{B}(t)] + \int_{0}^{t} \hat{\mathbf{K}}[\mathbf{B}(t), t-s](\mathbf{C}_t(s) - \mathbf{I})\, \mathrm{d}s \tag{8.11}$$

or

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \tilde{\mathbf{k}}[\mathbf{B}(t)] + \int_{0}^{t} \tilde{\mathbf{K}}[\mathbf{B}(t), t-s] \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} \mathrm{d}s. \tag{8.12}$$

$\hat{\mathbf{k}}[\mathbf{B}(t)]$ and $\tilde{\mathbf{k}}[\mathbf{B}(t)]$ have the same form as in (8.8). However, since $I_3(\mathbf{B}) = 1$, the scalar coefficients α_{i} now depend only on $I_1(\mathbf{B})$ and $I_2(\mathbf{B})$.

Lianis and co-workers carried out an extensive experimental program to determine $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{K}}$ for a styrene-butadiene rubber. The result of their program, summarized in [20], is the following specific form of (8.12),

$$
\begin{aligned}
\boldsymbol{\sigma}(t) = & -p\mathbf{I} + \left[a + \frac{b}{(I_1 - 2)^2} + I_1\left[c + \mathrm{d}\left(I_2 - 3\right)\right]\right]\mathbf{B}(t) - \left[c + \mathrm{d}\left(I_2 - 3\right)\right]\mathbf{B}(t)^2 \\
& + 2\int_{-\infty}^{t}\left[\phi_o\left(t - s\right) + \left(I_2 - 3\right)\Phi_o\left(t - s\right)\right]\frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s}\mathrm{d}s \\
& + \int_{-\infty}^{t}\left[\phi_1\left(t - s\right) + \frac{\bar{\phi}_1\left(t - s\right)}{(I_1 - 2)^2}\right]\left[\mathbf{B}(t)\frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} + \frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s}\mathbf{B}(t)\right]\mathrm{d}s. \qquad (8.13)
\end{aligned}
$$

In (8.13), I_1 and I_2 are the invariants of $\mathbf{B}$, a, b, c, d are constants and $\phi_o(t)$, $\Phi_o(t)$, $\phi_1(t)$, $\bar{\phi}_1(t)$ are monotonically decreasing functions of time. Table I of [20] lists values for these functions at a set of times, as well as for a, b, c, d.

It is interesting to quote the comment in [20] regarding (8.13) as it provides insight into the experimental effort required to determine a constitutive equation for a specific material: "Arriving at …(8.13) was a matter of trial and error, cross plotting, and curve fitting over a wide range of uniaxial and biaxial relaxation data. The fact that this equation predicts accurate results for other deformation histories is a matter of experimental verification."

8.4. Pipkin–Rogers Constitutive Theory

The tensor valued function $\mathbf{K}_3\,[\mathbf{C}, s]$ in (6.11) or (6.12) has the form

$$\mathbf{K}_3\,[\mathbf{C}, s] = \alpha_0\mathbf{I} + \alpha_1\mathbf{C} + \alpha_2\mathbf{C}^2, \qquad (8.14)$$

where $\alpha_0, \alpha_1, \alpha_2$ are functions of "s" and the invariants of $\mathbf{C}$,

$$I_1(\mathbf{C}) = \mathrm{tr}(\mathbf{C}), \quad I_2(\mathbf{C}) = \frac{1}{2}\left[\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right], \quad I_3(\mathbf{C}) = \det(\mathbf{C}). \qquad (8.15)$$

For notational convenience, let $I\,(\mathbf{C})$ denote the set $(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}))$. By (8.14) and the assumed dependence of $\mathbf{K}_3\,[\mathbf{C}, s]$ on "s", the scalar coefficients $\alpha_0, \alpha_1, \alpha_2$ also monotonically decrease with s to non-zero limit values.

For an isotropic material, (6.11) can be written as

$$\boldsymbol{\sigma}(t) = \mathbf{F}(t)\boldsymbol{\Pi}(t)\mathbf{F}(t)^T, \qquad (8.16)$$

where

$$
\begin{aligned}
\boldsymbol{\Pi}(t) = & \ \alpha_0(I(\mathbf{C}(t)), 0)\mathbf{I} + \alpha_1(I(\mathbf{C}(t)), 0)\mathbf{C}(t) + \alpha_2(I(\mathbf{C}(t)), 0)\mathbf{C}^2(t) \\
& + \int_0^t \frac{\partial}{\partial\,(t - s)}[\alpha_0(I(\mathbf{C}(s)), t - s)\mathbf{I} + \alpha_1(I(\mathbf{C}(s)), t - s)\mathbf{C}(s) \\
& + \alpha_2(I(\mathbf{C}(s)), t - s)\mathbf{C}^2(s)]\mathrm{d}s. \qquad (8.17)
\end{aligned}
$$

If the material is assumed to be incompressible, then (8.16) is modified by the addition of the term $-p\mathbf{I}$, as in the case of (6.12), and $I(\mathbf{C})$ now represents $(I_1(\mathbf{C}), I_2(\mathbf{C}))$ since deformations are restricted by (5.10) to satisfy the constraint $I_3(\mathbf{C}) = 1$. Recalling the assumption that the material is stress free in the reference configuration and hence $\mathbf{K}_3(\mathbf{I}, t) = 0$, it follows from (8.14) that

$$\alpha_0 \left(I(\mathbf{I}), s\right) + \alpha_1 \left(I(\mathbf{I}), s\right) + \alpha_2 \left(I(\mathbf{I}), s\right) = 0. \tag{8.18}$$

When the material is incompressible, $I(\mathbf{I}) = (3, 3, 1)$ and when it is incompressible $I(\mathbf{I}) = (3, 3)$.

The terms outside the integral can be expressed in terms of $\mathbf{B}(t)$ by use of (3.16) and the observation that $I_\alpha(\mathbf{C}) = I_\alpha(\mathbf{B})$, $\alpha = 1, 2, 3$. The integrand cannot be expressed in terms of $\mathbf{B}(t)$ because it depends on $\mathbf{C}(s)$ for all times $s \in [0, t]$. It is possible to express $\mathbf{F}(t)\mathbf{C}(s)\mathbf{F}(t)^T$ in terms of $\mathbf{B}(t)$ and $\mathbf{C}_t(s)$ by use of (3.13), (3.14) and (3.18). There seems to be no particular advantage in doing so and, therefore, it is not done here.

8.5. K-BKZ Constitutive theory

Kaye [21] and Bernstein, Kearsley and Zapas [22] proposed a constitutive equation for polymer fluids of the form

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \int_{-\infty}^{t} \left\{ \frac{\partial U\left(I_1, I_2, t-s\right)}{\partial I_1} \mathbf{C}_t(s)^{-1} - \frac{\partial U\left(I_1, I_2, t-s\right)}{\partial I_2} \mathbf{C}_t(s) \right\} \mathrm{d}s, \tag{8.19}$$

in which $U(I_1, I_2, s)$ is a material property that depends on time s and the scalar invariants of the relative right Cauchy–Green strain tensor $\mathbf{C}_t(s)$ introduced in (3.21),

$$I_1 = \operatorname{tr}\left(\mathbf{C}_t(s)^{-1}\right), \quad I_2 = \operatorname{tr}\left(\mathbf{C}_t(s)\right). \tag{8.20}$$

This constitutive equation, known as the K-BKZ model, is mentioned here for several reasons. It is a nonlinear single integral constitutive equation whose integrand is expressed in terms of finite strain tensors, just as in (8.6) or (8.17). In addition, with it a number of boundary value problems involving viscoelastic fluids can be approached in a manner similar to problems involving viscoelastic solids.

9. CONSTITUTIVE EQUATIONS FOR TRANSVERSELY ISOTROPIC AND ORTHOTROPIC MATERIALS

For many materials, such as biological tissue, it is appropriate to use a constitutive equation for an anisotropic viscoelastic solid. The restrictions due to transverse isotropy and orthotropy on the functions appearing in the rate and differential constitutive equations of Section 6.1, the Green–Rivlin constitutive equation of Section 6.2 and the finite linear viscoelastic constitutive equation of Section 6.3 lead to very complicated expressions. It is unlikely that they have appeared in the literature. In the case of the Pipkin–Rogers constitutive equation (6.11) or (6.12), the expressions are more tractable and have been discussed by Rajagopal and Wineman [23].

TRANSVERSE ISOTROPY

For a material that is transversely isotropic with respect to the $\mathbf{E}_3$ direction, the invariants are

$$\begin{aligned} I_1(\mathbf{C}) &= \mathrm{tr}(\mathbf{C}), \quad I_2(\mathbf{C}) = \frac{1}{2}\left[\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right], \quad I_3(\mathbf{C}) = \det(\mathbf{C}), \\ I_4(\mathbf{C}) &= C_{33}, \quad I_5(\mathbf{C}) = C_{13}^2 + C_{23}^2. \end{aligned} \tag{9.1}$$

Let $I(\mathbf{C})$ denote the set $(I_1(\mathbf{C}), \cdots, I_5(\mathbf{C}))$. (6.11) becomes

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathbf{F}(t)\{\alpha_1\ (I(\mathbf{C}(t),0)\mathbf{I} + \alpha_2(I(\mathbf{C}(t),0)\,[I_1\,(\mathbf{C}(t))\,I - \mathbf{C}(t)] + \alpha_3(I(\mathbf{C}(t),0)\mathbf{E}_3 \otimes \mathbf{E}_3 \\ &+ \alpha_4(I(\mathbf{C}(t),0)\,[C_{13}(t)\,(\mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1) + C_{23}(t)\,(\mathbf{E}_2 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_2)] \\ &+ \int_0^t \frac{\partial}{\partial\,(t-s)}\,[\alpha_1(I(\mathbf{C}(s)),t-s)\mathbf{I} + \alpha_2(I(\mathbf{C}(s)),t-s)\,(I_1\,(\mathbf{C}(s))\,I - \mathbf{C}(s)) \\ &+ \alpha_3(I(\mathbf{C}(s),t-s)\mathbf{E}_3 \otimes \mathbf{E}_3 + \alpha_4(I(\mathbf{C}(s),t-s)[C_{13}(s)\,(\mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1) \\ &+ C_{23}(s)\,(\mathbf{E}_2 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_2)]]\mathrm{d}s\}\mathbf{F}(t)^T \end{aligned} \tag{9.2}$$

ORTHOTROPY

For a material that is orthotropic, the invariants are

$$\begin{aligned} I_1(\mathbf{C}) &= C_{11}, \quad I_2(\mathbf{C}) = C_{22}, \quad I_3(\mathbf{C}) = C_{33}, \\ I_4(\mathbf{C}) &= C_{12}^2, \quad I_5(\mathbf{C}) = C_{23}^2, \quad I_6(\mathbf{C}) = C_{31}^2. \end{aligned} \tag{9.3}$$

Let $I(\mathbf{C})$ denote the set $(I_1(\mathbf{C}), \cdots, I_6(\mathbf{C}))$. (6.11) becomes

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathbf{F}(t)\{\alpha_1(I(\mathbf{C}),0)\ \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha_2(I(\mathbf{C}),0)\mathbf{E}_2 \otimes \mathbf{E}_2 + \alpha_3(I(\mathbf{C}),0)\mathbf{E}_3 \otimes \mathbf{E}_3 \\ &+ 2\alpha_4(I(\mathbf{C}),0)C_{12}\,(\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_1 \otimes \mathbf{E}_2) + 2\alpha_5(I(\mathbf{C}),0)C_{23}\,(\mathbf{E}_2 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_2) \\ &+ 2\alpha_6(I(\mathbf{C}),0)C_{31}\,(\mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1) \\ &+ \int_0^t \frac{\partial}{\partial\,(t-s)}\,[\alpha_1(I(\mathbf{C}(s)),t-s)\mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha_2(I(\mathbf{C}(s)),t-s)\mathbf{E}_2 \otimes \mathbf{E}_2 \\ &+ \alpha_3(I(\mathbf{C}(s)),t-s)\mathbf{E}_3 \otimes \mathbf{E}_3 + 2\alpha_4(I(\mathbf{C}(s)),t-s)C_{12}(s)\,(\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1) \\ &+ 2\alpha_5(I(\mathbf{C}(s)),t-s)C_{23}(s)\,(\mathbf{E}_2 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_2) \\ &+ 2\alpha_6(I(\mathbf{C}(s)),t-s)C_{31}(s)\,(\mathbf{E}_1 \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{E}_1)]\mathrm{d}s\}\mathbf{F}(t)^T. \end{aligned} \tag{9.4}$$

10. DEPENDENCE OF STRESS RELAXATION ON TEMPERATURE AND STRAIN—THE CLOCK CONCEPT

Material symmetry restrictions lead to explicit expressions for the tensorial structure of the constitutive equations, that is, it shows how the stress tensor is expressed in terms of the deformation tensor. This section is concerned with factors that influence the time dependence of response.

It is well established that stress relaxation is affected by temperature [6, Chapter 11]. This effect is accounted for by introducing a new time variable ξ, called the material time, intrinsic time or reduced time. It arises from the notion that during stress relaxation, the macromolecular structure goes through a sequence of reconfigurations. ξ represents the time during this sequence as seen by the material and differs from the laboratory time t. That is, the material follows its own "clock" for stress relaxation that can run faster or slower than the laboratory clock.

The increment of material time $\mathrm{d}\xi$ is related to the increment of laboratory time $\mathrm{d}t$ by

$$\mathrm{d}\xi = \frac{\mathrm{d}t}{a\left(T(t), T_o\right)}, \tag{10.1}$$

in which T_o is a reference temperature, $T(t)$ is the current temperature, and $a\left(T(t), T_o\right)$ is a material property called the *time-temperature shift function*. $a\left(T(t), T_o\right) > 0$, thereby ensuring that $\mathrm{d}\xi > 0$. When $T(t) > T_o$, $a\left(T(t), T_o\right) < 1$ and $\mathrm{d}\xi/\mathrm{d}t > 1$ so that the material time increment is larger than the laboratory time increment (the material clock moves faster). When $T(t) < T_o$, $a\left(T(t), T_o\right) > 1$ and $\mathrm{d}\xi/\mathrm{d}t < 1$. The material time increment is now smaller than the laboratory time increment (the material clock moves slower). The current material time ξ is related to the current laboratory time by

$$\xi(t) = \int_0^t \frac{\mathrm{d}x}{a\left(T(x), T_o\right)}, \tag{10.2}$$

a relation often referred to as defining a "temperature" clock.

There has been recent experimental evidence that the time dependence of response is also affected by strain. This has led to the notion of a "strain" clock, whose definition is analogous to the "temperature" clock,

$$\xi(t) = \int_0^t \frac{\mathrm{d}x}{\hat{a}\left(\mathbf{C}(x)\right)}. \tag{10.3}$$

Knauss and Emri [24] and Shay and Caruthers [25], based on considerations from polymer science, assumed that $\hat{a}\left(\mathbf{C}(x)\right)$ depends on the volumetric strain. McKenna and Zapas [26] interpreted results on torsion of PMMA as indicating that $\hat{a}\left(\mathbf{C}(x)\right)$ should depend on shear stain. Experiments by Liechti and Popelar [27] led them to express $\hat{a}\left(\mathbf{C}(x)\right)$ in terms of both volumetric and shear strains. Recently, Caruthers et al. [28] proposed a constitutive theory that expresses $\hat{a}\left(\mathbf{C}(x)\right)$ in terms of the "configurational energy" of the molecular structure. This led to an expression for the $\hat{a}\left(\mathbf{C}(x)\right)$ in terms of the history of $\mathbf{C}(z)$, $z \in [0, x]$. The "strain" clock concept is a subject of ongoing research.

The clock is introduced in to the constitutive equation by replacing the constitutive assumption (5.1) with

$$\boldsymbol{\sigma}(t) = \mathcal{F}\left[\mathbf{F}(\xi(t) - \xi(s))|_{s=0}^{t}\right]. \tag{10.4}$$

The restrictions due to superposed rigid body motions and material symmetry can be imposed on constitutive equations of the form (10.4). The dependence of $\hat{a}\,(\mathbf{C}(x))$ on $\mathbf{C}$ arises from the former restriction. The latter restriction implies that $\hat{a}\,(\mathbf{C}(x))$ depends on the appropriate invariants of $\mathbf{C}$. The Green-Rivlin, finite linear viscoelasticity and Pipkin–Rogers models can still be obtained by appropriate assumptions, but with the argument $t - s$ replaced by $\xi(t) - \xi(s)$. These expressions are not given here for the sake of brevity.

11. VOLTERRA INTEGRAL EQUATIONS

The remainder of this article is devoted to examples using the constitutive equations for isotropic materials given by finite linear viscoelasticity and the Pipkin–Rogers theories, both referred to as nonlinear single integral constitutive equations. As will be seen in later sections, their application to boundary value problems leads to equations of the form

$$\mathbf{f}(t) = \boldsymbol{\Phi}_1\,(\mathbf{x}(t)) + \int_0^t \boldsymbol{\Phi}_2\,(\mathbf{x}(t), \mathbf{x}(s), t - s)\,\mathrm{d}s, \tag{11.1}$$

where $\mathbf{f}(t)$ is a known vector-valued function of t and $\mathbf{x}(t)$ is an unknown function of t. $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are known vector valued functions of their arguments. Equation (11.1) is said to be a Volterra integral equation for $\mathbf{x}(t)$ given $\mathbf{f}(t)$.

The special case of linear Volterra integral equations occurs when $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are linear in $\mathbf{x}(t)$ and $\mathbf{x}(s)$. Such equations have already been encountered in the discussions following (2.18) an (2.25). Issues such as existence and uniqueness of solutions and analytical methods for finding them are discussed in [29] and [30]. Linear Volterra integral equations of the type that occur in linear viscoelasticity can generally be solved by analytical methods such as successive approximations or the Laplace transform. However, very few, if any, analytical methods have been developed for nonlinear Volterra integral equations, even those that occur in nonlinear viscoelasticity. Even so, in the case of linear Volterra integral equations, analytical methods lead to expressions for the solution that require substantial numerical effort to evaluate. Because of this, the most useful method of solving Volterra integral equations, whether linear or nonlinear, is numerical. This was shown to be the case in linear viscoelasticity in by Lee and Rogers [31, 32] and Lee et al. [33]. This is also the case for nonlinear viscoelasticity, as will be discussed in Section 18.

For this reason, the remainder of his section is devoted to the description of a numerical method of solution of (11.1). There are other reasons why a numerical method of solution is presented here. First, an analytical solution, if available, would provide a formula for $\mathbf{x}(t)$. One might see the $\mathbf{x}(t)$ curve as a single entity and miss the visual sense of the solution evolving as time increases, i.e., the sense of seeing the solution evolve as if produced by a cursor moving across a computer screen. The method of numerical solution helps one to visualize this evolution. The second reason for presenting a numerical method of solution is that it will be useful when discussing applications in later sections.

Equation (11.1) is to be satisfied at a discrete set of times denoted by t_n, $n = 1, \ldots, n_{\max}$, with $t_1 = 0$. For notational convenience, let $\mathbf{f}(t_\mathrm{i}) = \mathbf{f}_\mathrm{i}$ and $\mathbf{x}(t_\mathrm{i}) = \mathbf{x}_\mathrm{i}$. Then, (11.1) can be restated as

$$\mathbf{f}_n = \boldsymbol{\Phi}_1\left(\mathbf{x}_n\right) + \int_{t_1}^{t_n} \boldsymbol{\Phi}_2\left(\mathbf{x}_n, \mathbf{x}(s), t_n - s\right) \mathrm{d}s. \tag{11.2}$$

When $n = 1$, (11.2) reduces to

$$\mathbf{f}_1 = \boldsymbol{\Phi}_1\left(\mathbf{x}_1\right), \tag{11.3}$$

a nonlinear equation for $\mathbf{x}_1$. There are many methods for solving such equations. Assuming that $\mathbf{x}_1$ has been found, (11.2) must now be satisfied at time t_2. Thus, consider

$$\mathbf{f}_2 = \boldsymbol{\Phi}_1\left(\mathbf{x}_2\right) + \int_{t_1}^{t_2} \boldsymbol{\Phi}_2\left(\mathbf{x}_2, \mathbf{x}(s), t_2 - s\right) \mathrm{d}s. \tag{11.4}$$

Let the integral be approximated using the trapezoidal rule,

$$\mathbf{f}_2 = \boldsymbol{\Phi}_1\left(\mathbf{x}_2\right) + \frac{1}{2}\left(t_2 - t_1\right)\left[\boldsymbol{\Phi}_2\left(\mathbf{x}_2, \mathbf{x}_2, 0\right) + \boldsymbol{\Phi}_2\left(\mathbf{x}_2, \mathbf{x}_1, t_2 - t_1\right)\right]. \tag{11.5}$$

Since $\mathbf{x}_1$ has been found, (11.5) is a nonlinear equation for $\mathbf{x}_2$. Next, let $n \geq 2$ and assume that solutions $\mathbf{x}_\mathrm{i}$, $\mathrm{i} = 1, \ldots, n-1$ have been found. At time t_n, (11.2) can be written as

$$\mathbf{f}_n = \boldsymbol{\Phi}_1\left(\mathbf{x}_n\right) + \sum_{\mathrm{i}=1}^{\mathrm{i}=n-1} \int_{t_\mathrm{i}}^{t_{\mathrm{i}+1}} \boldsymbol{\Phi}_2\left(\mathbf{x}_n, \mathbf{x}(s), t_n - s\right) \mathrm{d}s. \tag{11.6}$$

As before, each integral in (11.6) is approximated using the trapezoidal rule,

$$\mathbf{f}_n = \boldsymbol{\Phi}_1\left(\mathbf{x}_n\right) + \sum_{\mathrm{i}=1}^{\mathrm{i}=n-1} \frac{1}{2}\left(t_{\mathrm{i}+1} - t_\mathrm{i}\right)\left[\boldsymbol{\Phi}_2\left(\mathbf{x}_n, \mathbf{x}_{\mathrm{i}+1}, t_n - t_{\mathrm{i}+1}\right) + \boldsymbol{\Phi}_2\left(\mathbf{x}_n, \mathbf{x}_\mathrm{i}, t_n - t_\mathrm{i}\right)\right]. \tag{11.7}$$

This gives a nonlinear equation for $\mathbf{x}_n$. One continues in this manner, producing the solution at each time t_n as time marches forward until $n = n_{\max}$.

The basic numerical method outlined here can be refined in many ways. For details, see [29] or [31].

12. HOMOGENEOUS DEFORMATIONS—STEP CHANGE HISTORIES

Let a nonlinear viscoelastic solid undergo a homogeneous deformation in which there is a step change from the reference configuration to fixed deformed configuration. This is a special case of the motion (3.1) and is described by

$$\begin{aligned} \mathbf{x}(s) &= \mathbf{X}, \quad s \in (-\infty, 0) \\ \mathbf{x}(s) &= \mathbf{F}_o\mathbf{X}, \quad s \in [0, t]\,, \end{aligned} \tag{12.1}$$

where $\mathbf{F}_o$ is constant. By (3.8), the deformation gradient history is

$$\begin{aligned} \mathbf{F}(s) &= \mathbf{I}, \quad s \in (-\infty, 0) \\ \mathbf{F}(s) &= \mathbf{F}_o, \quad s \in [0, t]\,. \end{aligned} \tag{12.2}$$

Then, by (3.15),

$$\begin{aligned} \mathbf{C}(s) &= \mathbf{I}, \quad s \in (-\infty, 0) \\ \mathbf{C}(s) &= \mathbf{F}_o^T\mathbf{F}_o, \\ &\equiv \mathbf{C}_o, \quad s \in [0, t]\,, \end{aligned} \tag{12.3}$$

and by (3.16)

$$\mathbf{B}(t) = \mathbf{F}_o\mathbf{F}_o^T \equiv \mathbf{B}_o. \tag{12.4}$$

The relative stretch history is found from (3.17) and (12.2),

$$\begin{aligned} \mathbf{F}_t(s) &= \mathbf{F}(s)\mathbf{F}(t)^{-1} \\ &= \mathbf{F}_o^{-1}, \quad s \in (-\infty, 0) \\ &= \mathbf{I}, \quad s \in [0, t]\,. \end{aligned} \tag{12.5}$$

Then, using (3.21) and (12.4),

$$\begin{aligned} \mathbf{C}_t(s) &= \mathbf{F}_o^{-T}\mathbf{F}_o^{-1} \\ &\equiv \mathbf{B}_o^{-1}, \quad s \in (-\infty, 0) \\ &= \mathbf{I}, \quad s \in [0, t]\,. \end{aligned} \tag{12.6}$$

FINITE LINEAR VISCOELASTICITY

Let $\mathbf{C}(s) = \mathbf{C}_o$ and the history of $\mathbf{C}_t(s)$ from (12.6) be introduced into (6.9_2),

$$\boldsymbol{\sigma}(t) = \mathbf{F}_o \left\{ \mathbf{k}_2 [\mathbf{C}_o] + \int_{-\infty}^{0} \mathbf{K}_2 [\mathbf{C}_o, t-s] \left[\mathbf{F}_o^T \left(\mathbf{B}_o^{-1} - \mathbf{I} \right) \mathbf{F}_o \right] \mathrm{d}s \right\} \mathbf{F}_o^T. \tag{12.7}$$

On letting $x = t - s$, Equation (12.7) becomes

$$\boldsymbol{\sigma}(t) = \mathbf{F}_o \left\{ \mathbf{k}_2 [\mathbf{C}_o] + \int_{t}^{\infty} \mathbf{K}_2 [\mathbf{C}_o, x] \, \mathrm{d}x \cdot \left[\mathbf{F}_o^T \left(\mathbf{B}_o^{-1} - \mathbf{I} \right) \mathbf{F}_o \right] \right\} \mathbf{F}_o^T. \tag{12.8}$$

According to the theory of Coleman and Noll [15], $\mathbf{K}_2 [\mathbf{C}_o, x]$ decays monotonically to zero sufficiently fast as x increases so as to ensure the convergence of the integral in (12.8). Thus, as $t \to \infty$, the integral term decreases to zero. This discussion can be extended to the incompressible case (6.10_2) by replacing $\boldsymbol{\sigma}(t)$ in (12.8) by $\boldsymbol{\sigma}(t) + p\mathbf{I}$.

Consider the constitutive equation for the isotropic material given by (8.7) and (8.10). By (12.4), $\mathbf{B}(t) = \mathbf{B}_o$. In addition, because $\mathbf{C}_t(s)$ has the jump discontinuity indicated in (12.7),

$$\frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} = \left(\mathbf{I} - \mathbf{B}_o^{-1} \right) \delta(s), \tag{12.9}$$

where $\delta(s)$ denotes the Dirac delta function. The stress relaxation response is

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B}_o + \alpha_2 \mathbf{B}_o^2 + 2 \sum_{\alpha=0}^{2} \phi_\alpha (t-s) \left[\mathbf{B}_o^\alpha - \mathbf{B}_o^{\alpha-1} \right] \\ &\quad + \sum_{\alpha=0}^{2} \sum_{\beta=0}^{2} \phi_{\alpha\beta}(t-s) \mathbf{B}_o^\alpha \mathrm{tr} \left[\mathbf{B}_o^\beta - \mathbf{B}_o^{\beta-1} \right]. \end{aligned} \tag{12.10}$$

The scalar coefficients α_{i}, ϕ_α and $\phi_{\alpha\beta}$ are functions of the invariants $I_\alpha(\mathbf{B}_o)$ of $\mathbf{B}_o$.

This is extended to the incompressible case (6.10_2) by replacing $\boldsymbol{\sigma}(t)$ in (12.10) by $\boldsymbol{\sigma}(t) + p\mathbf{I}$. The special case of the Lianis constitutive equation (8.13) becomes

$$\begin{aligned} \boldsymbol{\sigma}(t) &= -p\mathbf{I} + \left[a + 2\phi_1(t) + \frac{b + 2\bar{\phi}_1(t)}{(I_1 - 2)^2} + I_1 \left[c + 2\phi_o(t) + (I_2 - 3)\left(\mathrm{d} + 2\Phi_o(t)\right) \right] \right] \mathbf{B}_o \\ &\quad - \left[c + 2\phi_o + (I_2 - 3)\left(\mathrm{d} + 2\Phi_o(t)\right) \right] \mathbf{B}_o^2, \end{aligned} \tag{12.11}$$

where I_1 and I_2 are the invariants of $\mathbf{B}_o$. In deriving (12.11), use was made of the Cayley–Hamilton theorem in the form

$$\mathbf{B}_o^{-1} = \mathbf{B}_o^2 - I_1 \mathbf{B}_o + I_2 \mathbf{I}. \tag{12.12}$$

PIPKIN–ROGERS THEORY

Let the histories of $\mathbf{F}(s)$ in (12.2) and $\mathbf{C}(s)$ in (12.3) be substituted into the Pipkin–Rogers constitutive equation (6.11),

$$\boldsymbol{\sigma}(t) = \mathbf{F}_o \left\{ \mathbf{K}_3 \left[\mathbf{C}_o, 0\right] + \int_0^t \frac{\partial}{\partial (t-s)} \mathbf{K}_3 \left[\mathbf{C}_o, t-s\right] \mathrm{d}s \right\} \mathbf{F}_o^T. \tag{12.13}$$

On noting that $\mathbf{C}_o$ is constant, letting $x = t - s$, and then integrating by parts, one obtains

$$\boldsymbol{\sigma}(t) = \mathbf{F}_o \mathbf{K}_3 \left[\mathbf{C}_o, t\right] \mathbf{F}_o^T, \tag{12.14}$$

This is a deformation dependent stress relaxation function. The incompressible case (6.11) is obtained by replacing $\boldsymbol{\sigma}(t)$ in (12.14) by $\boldsymbol{\sigma}(t) + p\mathbf{I}$. When the material is isotropic, $\mathbf{K}_3 \left[\mathbf{C}_o, t\right]$ is given by (8.14). Aside from the argument t in the coefficients $\alpha_o, \alpha_1, \alpha_2$ in (8.14), this has the same form of dependence on $\mathbf{F}_o$ and $\mathbf{C}_o$ as in the constitutive equation for nonlinear elasticity. The right-hand side of (12.14) can be expressed in terms of $\mathbf{B}_o$ using (12.3) and (12.4), but will not be done here.

Equations (12.8) and (12.10) have the form

$$\boldsymbol{\sigma}(t) = \mathbf{g}_1 \left(\mathbf{F}_o\right) + \mathbf{g}_2 \left(\mathbf{F}_o, t\right), \tag{12.15}$$

while (12.11) has the form

$$\boldsymbol{\sigma}(t) + p\mathbf{I} = \hat{\mathbf{g}}_1 \left(\mathbf{F}_o\right) + \hat{\mathbf{g}}_2 \left(\mathbf{F}_o, t\right). \tag{12.16}$$

A comparison of (12.15) and (2.24$_1$) shows that G_∞ corresponds to $\mathbf{g}_1(\mathbf{F}_o)$ and $\Delta G(t)$ corresponds to $\mathbf{g}_2(\mathbf{F}_o, t)$. The terms in (12.8) corresponding to $\mathbf{g}_2(\mathbf{F}_o, t)$ monotonically decrease to zero as $t \to \infty$ for reasons already mentioned. The terms corresponding to $\mathbf{g}_2(\mathbf{F}_o, t)$ in (12.10) and by $\hat{\mathbf{g}}_2(\mathbf{F}_o, t)$ in (12.11) monotonically decrease to zero as $t \to \infty$ because $\phi_\alpha(I(\mathbf{B}_o), t)$ and $\phi_{\alpha\beta}(I(\mathbf{B}_o), t)$ do so in (12.10) and $\phi_o(t)$, $\Phi_o(t)$, $\phi_1(t)$ and $\bar{\phi}_1(t)$ do so in (12.11). The expressions for $\boldsymbol{\sigma}(t)$ in (12.8), (12.10) and (12.14) and for $\boldsymbol{\sigma}(t) + p\mathbf{I}$ in (12.11) are therefore deformation dependent stress relaxation functions and are generalizations to three dimensions of $G(t, \varepsilon_o)$ introduced in Section 2.2.

13. HOMOGENEOUS DEFORMATIONS—TRIAXIAL STRETCH HISTORIES

Let an isotropic nonlinear viscoelastic solid block undergo the triaxial stretch motion

$$x_\mathrm{i}(s) = \lambda_\mathrm{i}(s) X_\mathrm{i}, \quad s \in (-\infty, t], \quad \mathrm{i} = 1, 2, 3$$

with

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = 1, \quad s \in (-\infty, 0). \tag{13.1}$$

$\lambda_{\mathrm{i}}(s)$ is a stretch ratio in the X_{i} direction. It can have a jump discontinuity at $s = 0$ and then vary arbitrarily for $s \in [0, t]$. By (3.8), the deformation gradient history is

$$\mathbf{F}(s) = \mathbf{I}, \quad s \in (-\infty, 0)$$
$$\mathbf{F}(s) = \operatorname{diag}\left[\lambda_1(s), \lambda_2(s), \lambda_3(s)\right], \quad s \in [0, t], \tag{13.2}$$

from which the kinematical quantities needed for use in the constitutive equations are found to be given by

$$\mathbf{C}(s) = \mathbf{I}, \quad s \in (-\infty, 0)$$
$$\mathbf{C}(s) = \operatorname{diag}\left[\lambda_1(s)^2, \lambda_2(s)^2, \lambda_3(s)^2\right], \quad s \in [0, t], \tag{13.3}$$
$$\mathbf{B}(t) = \operatorname{diag}\left[\lambda_1(t)^2, \lambda_2(t)^2, \lambda_3(t)^2\right], \tag{13.4}$$
$$\mathbf{C}_t(s) = \operatorname{diag}\left[1/\lambda_1(t)^2, 1/\lambda_2(t)^2, 1/\lambda_3(t)^2\right], \quad s \in (-\infty, 0)$$
$$\mathbf{C}_t(s) = \operatorname{diag}\left[\left(\lambda_1(s)/\lambda_1(t)\right)^2, \left(\lambda_2(s)/\lambda_2(t)\right)^2, \left(\lambda_3(s)/\lambda_3(t)\right)^2\right], \quad s \in [0, t]. \tag{13.5}$$

In addition,

$$I_1\left(\mathbf{C}(s)\right) = \lambda_1(s)^2 + \lambda_2(s)^2 + \lambda_3(s)^2,$$
$$I_2\left(\mathbf{C}(s)\right) = \lambda_1(s)^2\lambda_2(s)^2 + \lambda_2(s)^2\lambda_3(s)^2 + \lambda_3(s)^2\lambda_1(s)^2,$$
$$I_3\left(\mathbf{C}(s)\right) = \lambda_1(s)^2\lambda_2(s)^2\lambda_3(s)^2. \tag{13.6}$$

Note that $I_k\left(\mathbf{B}(t)\right) = I_k\left(\mathbf{C}(t)\right), k = 1, 2, 3$. Finally,

$$\frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} = \mathbf{0}, \quad s \in (-\infty, 0),$$
$$\frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} = \operatorname{diag}\left[\left(\frac{\lambda_1(0)}{\lambda_1(t)}\right)^2 - \left(\frac{1}{\lambda_1(t)}\right)^2, \left(\frac{\lambda_2(0)}{\lambda_2(t)}\right)^2 - \left(\frac{1}{\lambda_2(t)}\right)^2, \left(\frac{\lambda_3(0)}{\lambda_3(t)}\right)^2 - \left(\frac{1}{\lambda_3(t)}\right)^2\right]\delta(s)$$
$$\frac{\mathrm{d}\mathbf{C}_t(s)}{\mathrm{d}s} = \operatorname{diag}\left[2\frac{\lambda_1(s)}{\lambda_1(t)^2}\frac{\mathrm{d}\lambda_1(s)}{\mathrm{d}s}, 2\frac{\lambda_2(s)}{\lambda_2(t)^2}\frac{\mathrm{d}\lambda_2(s)}{\mathrm{d}s}, 2\frac{\lambda_3(s)}{\lambda_3(t)^2}\frac{\mathrm{d}\lambda_3(s)}{\mathrm{d}s}\right], \quad s \in [0, t]. \tag{13.7}$$

ISOTROPIC FINITE LINEAR VISCOELASTICITY

Substituting (13.4)–(13.7) into (8.7)–(8.10) gives an expression for the stress $\sigma_{\mathrm{ii}}(t)$,

$$\begin{aligned}\sigma_{\mathrm{ii}}(t) =\ & \alpha_o + \alpha_1\lambda_{\mathrm{i}}(t)^2 + \alpha_2\lambda_{\mathrm{i}}(t)^4 + 2\sum_{\alpha=0}^{2}\phi_\alpha(t)\left[\lambda_{\mathrm{i}}(t)^{2\alpha}\left[\left(\frac{\lambda_{\mathrm{i}}(0)}{\lambda_{\mathrm{i}}(t)}\right)^2 - \left(\frac{1}{\lambda_{\mathrm{i}}(t)}\right)^2\right]\right] \\ & + 4\int_0^t \sum_{\alpha=0}^{2}\phi_\alpha\,(t-s)\,\lambda_{\mathrm{i}}(t)^{2\alpha}\frac{\lambda_{\mathrm{i}}(s)}{\lambda_{\mathrm{i}}(t)^2}\frac{\mathrm{d}\lambda_{\mathrm{i}}(s)}{\mathrm{d}s}\mathrm{d}s \\ & + 2\sum_{\alpha=0}^{2}\sum_{\beta=0}^{2}\phi_{\alpha\beta}(t)\lambda_{\mathrm{i}}(t)^{2\alpha}\sum_{k=1}^{3}\left[\lambda_k(t)^{2\beta}\left[\left(\frac{\lambda_k(0)}{\lambda_k(t)}\right)^2 - \left(\frac{1}{\lambda_k(t)}\right)^2\right]\right] \\ & + 4\int_0^t \sum_{\alpha=0}^{2}\sum_{\beta=0}^{2}\phi_{\alpha\beta}\,(t-s)\,\lambda_{\mathrm{i}}(t)^{2\alpha}\left(\sum_{k=1}^{3}\lambda_k(t)^{2\beta}\frac{\lambda_k(s)}{\lambda_k(t)^2}\frac{\mathrm{d}\lambda_k(s)}{\mathrm{d}s}\right)\mathrm{d}s. \qquad (13.8)\end{aligned}$$

For an incompressible material, $\sigma_{\mathrm{ii}}(t)$ is replaced by $\sigma_{\mathrm{ii}}(t) + p$ and the motion is such that $\lambda_1(s)\lambda_2(s)\lambda_3(s) = 1$, $s \in [0, t]$.

For the Lianis constitutive equation (8.13),

$$\begin{aligned}\sigma_{\mathrm{ii}}(t) =\ & -p + \left[a + \frac{b}{(I_1-2)^2} + I_1\,(I_2-3)\right]\lambda_{\mathrm{i}}(t)^2 - \left[c + \mathrm{d}\,(I_2-3)\right]\lambda_{\mathrm{i}}(t)^4 \\ & + 2\left[\phi_o(t) + (I_2-3)\,\Phi_o(t)\right]\left[\left(\frac{\lambda_{\mathrm{i}}(0)}{\lambda_{\mathrm{i}}(t)}\right)^2 - \left(\frac{1}{\lambda_{\mathrm{i}}(t)}\right)^2\right] \\ & + 4\int_0^t \left[\phi_o(t-s) + (I_2-3)\,\Phi_o(t-s)\right]\lambda_{\mathrm{i}}(s)\frac{\lambda_{\mathrm{i}}(s)}{\lambda_{\mathrm{i}}(t)^2}\frac{\mathrm{d}\lambda_{\mathrm{i}}(s)}{\mathrm{d}s}\mathrm{d}s \\ & + 4\left[\phi_1(t) + \frac{\bar{\phi}_1(t)}{(I_1-2)^2}\right]\left[\left(\frac{\lambda_{\mathrm{i}}(0)}{\lambda_{\mathrm{i}}(t)}\right)^2 - \left(\frac{1}{\lambda_{\mathrm{i}}(t)}\right)^2\right]\lambda_{\mathrm{i}}(t)^2 \\ & + 4\int_0^t \left[\phi_1(t-s) + \frac{\bar{\phi}_1(t-s)}{(I_1-2)^2}\right]\lambda_{\mathrm{i}}(t)^2\lambda_{\mathrm{i}}(s)\frac{\lambda_{\mathrm{i}}(s)}{\lambda_{\mathrm{i}}(t)^2}\frac{\mathrm{d}\lambda_{\mathrm{i}}(s)}{\mathrm{d}s}s. \qquad (13.9)\end{aligned}$$

PIPKIN–ROGERS CONSTITUTIVE EQUATION

From (8.16),

$$\begin{aligned}\sigma_{\mathrm{ii}}(t) =\ & \lambda_{\mathrm{i}}(t)^2\left\{\alpha_0\left(I\left(\mathbf{C}(t)\right),0\right) + \alpha_1\left(I\left(\mathbf{C}(t)\right),0\right)\lambda_{\mathrm{i}}(t)^2 + \alpha_2\left(I\left(\mathbf{C}(t)\right),0\right)\lambda_{\mathrm{i}}(t)^4\right. \\ & + \int_0^t \frac{\partial}{\partial\,(t-s)}[\alpha_0\left(I\left(\mathbf{C}(s)\right),t-s\right) + \alpha_1\left(I\left(\mathbf{C}(s)\right),t-s\right)\lambda_{\mathrm{i}}(s)^2 \\ & \left. + \ \alpha_2\left(I\left(\mathbf{C}(s)\right),t-s\right)\lambda_{\mathrm{i}}(s)^4]\mathrm{d}s\right\}, \qquad (13.10)\end{aligned}$$

in which $I(\mathbf{C})$ denotes the set of invariants (13.6). For an incompressible material, $\sigma_{\mathrm{ii}}(t)$ is replaced by $\sigma_{\mathrm{ii}}(t)+p$, the motion is such that $\lambda_1(s)\lambda_2(s)\lambda_3(s)=1$, $s\in[0,t]$ or $I_3(\mathbf{C}(s))=1$ and $I(\mathbf{C})=(I_1(\mathbf{C}),I_2(\mathbf{C}))$.

In the sections to follow, analyses and discussions are presented using the Pipkin–Rogers constitutive theory because of its convenience for developing certain results. A corresponding treatment could also be carried using the constitutive equation for finite linear viscoelasticity, but is not done so because of space limitations. It is important to point out that the latter constitutive equation has been used in numerous applications that will be mentioned in due course.

14. HOMOGENEOUS DEFORMATIONS—UNIAXIAL STRETCH HISTORIES

Uniaxial stretch is the special case of triaxial stretch when there is only one non-zero stress component. As in other areas of solid mechanics such as elasticity and plasticity, an understanding of uniaxial stretch is essential to an understanding of the material. Thus, this section contains a detailed discussion of uniaxial stretch for nonlinear viscoelasticity. Many of the features of the uniaxial response introduced in Section 2 for linear viscoelastic response are re-visited here.

Let the reference configuration of an isotropic nonlinear viscoelastic solid be a block with edges along the X_1, X_2, X_3 axes of a cartesian coordinate system. The block undergoes uniaxial extension along the X_3-axis. The motion is described by (13.1) and the stresses are given by (13.10) with $\sigma_{11}(t)=\sigma_{22}(t)=0$, $t>0$. For notational convenience, let $\lambda_3(t)=\lambda(t)$ and $\sigma_{33}(t)=\sigma(t)$. Equations (13.10) become

$$\begin{aligned}\sigma(t) =& -p+\alpha_0(I(\mathbf{C}(t)),0)\,\lambda(t)^2+\alpha_1(I(\mathbf{C}(t)),0)\,\lambda(t)^4+\alpha_2(I(\mathbf{C}(t)),0)\,\lambda(t)^6\\ &+\int_0^t\frac{\partial}{\partial(t-s)}\left[\alpha_0(I(\mathbf{C}(s)),t-s)\,\lambda(t)^2+\alpha_1(I(\mathbf{C}(s)),t-s)\,\lambda(t)^2\lambda(s)^2\right.\\ &\left.+\ \alpha_2(I(\mathbf{C}(s)),t-s)\,\lambda(t)^2\lambda(s)^4\right]\mathrm{d}s, \end{aligned} \tag{14.1$_1$}$$

$$\begin{aligned}0 =& -p+\alpha_0(I(\mathbf{C}(t)),0)\,\lambda_1(t)^2+\alpha_1(I(\mathbf{C}(t)),0)\,\lambda_1(t)^4+\alpha_2(I(\mathbf{C}(t)),0)\,\lambda_1(t)^6\\ &+\int_0^t\frac{\partial}{\partial(t-s)}\left[\alpha_0(I(\mathbf{C}(s)),t-s)\,\lambda_1(t)^2+\alpha_1(I(\mathbf{C}(s)),t-s)\,\lambda_1(t)^2\lambda_1(s)^2\right.\\ &\left.+\ \alpha_2(I(\mathbf{C}(s)),t-s)\,\lambda_1(t)^2\lambda_1(s)^4\right]\mathrm{d}s, \end{aligned} \tag{14.1$_2$}$$

$$\begin{aligned}0 =& -p+\alpha_0(I(\mathbf{C}(t)),0)\,\lambda_2(t)^2+\alpha_1(I(\mathbf{C}(t)),0)\,\lambda_2(t)^4+\alpha_2(I(\mathbf{C}(t)),0)\,\lambda_2(t)^6\\ &+\int_0^t\frac{\partial}{\partial(t-s)}\left[\alpha_0(I(\mathbf{C}(s)),t-s)\,\lambda_2(t)^2+\alpha_1(I(\mathbf{C}(s)),t-s)\,\lambda_2(t)^2\lambda_2(s)^2\right.\\ &\left.+\ \alpha_2(I(\mathbf{C}(s)),t-s)\,\lambda_2(t)^2\lambda_2(s)^4\right]\mathrm{d}s. \end{aligned} \tag{14.1$_3$}$$

It is intended that ($14.1_{1,2,3}$) apply to both compressible and incompressible materials in a single expression. When the material is compressible, $p = 0$ and when it is incompressible Equations ($14.1_{1,2,3}$) are supplemented by the condition $\lambda_1(s)\lambda_2(s)\lambda_3(s) = 1$, $s \in [0, t]$.

Suppose that the material is compressible. If the history $\lambda_3(s) = \lambda(s)$, $s \in [0, t]$ is specified, then ($14.1_{2,3}$) become a system of nonlinear Volterra integral equations for $\lambda_1(s)$ and $\lambda_2(s)$, $s \in [0, t]$. Once these are known, (14.1_1) is used to determine $\sigma(t)$, $t > 0$. If the stress history, $\sigma(t)$, $t > 0$, is specified, ($14.1_{1,2,3}$) becomes a system of nonlinear Volterra integral equations for $\lambda_1(s)$, $\lambda_2(s)$, $\lambda_3(s)$, $s \in [0, t]$.

Suppose that the material is incompressible. If the history $\lambda_3(s) = \lambda(s)$, $s \in [0, t]$ is specified, then ($14.1_{2,3}$) along with $\lambda_1(s)\lambda_2(s)\lambda_3(s) = 1$, $s \in [0, t]$ become a system of nonlinear Volterra integral equations for p, $\lambda_1(s)$ and $\lambda_2(s)$, $s \in [0, t]$. Once these are known, (14.1_1) is used to determine $\sigma(t)$, $t > 0$. If the stress history, $\sigma(t)$, $t > 0$, is specified, ($14.1_{1,2,3}$) along with $\lambda_1(s)\lambda_2(s)\lambda_3(s) = 1$, $s \in [0, t]$ becomes a system of nonlinear Volterra integral equations for p, $\lambda_1(s)$, $\lambda_2(s)$, $\lambda_3(s)$, $s \in [0, t]$. These can be solved using the numerical method outlined in Section 11.

14.1. Determination of $\lambda_1(s)$ *and* $\lambda_2(s)$*,* $s \in [0, t]$*.*

For both compressible and incompressible materials, subtraction of (14.1_2) and (14.1_3) gives

$$\begin{aligned} 0 = {} & \left(\lambda_2(t)^2 - \lambda_1(t)^2\right)\left[\alpha_0\left(I\left(\mathbf{C}(t)\right),0\right) + \alpha_1\left(I\left(\mathbf{C}(t)\right),0\right)\left(\lambda_2(t)^2 + \lambda_1(t)^2\right)\right. \\ & \left. +\ \alpha_2\left(I\left(\mathbf{C}(t)\right),0\right)\left(\lambda_2(t)^4 + \lambda_2(t)^2\lambda_1(t)^2 + \lambda_1(t)^4\right)\right] \\ & + \int_0^t \frac{\partial}{\partial\left(t-s\right)}\left[\alpha_0\left(I\left(\mathbf{C}(s)\right),t-s\right)\left(\lambda_2(t)^2 - \lambda_1(t)^2\right)\right. \\ & +\ \alpha_1\left(I\left(\mathbf{C}(s)\right),t-s\right)\left(\lambda_2(t)^2\lambda_2(s)^2 - \lambda_1(t)^2\lambda_1(s)^2\right) \\ & \left. +\ \alpha_2\left(I\left(\mathbf{C}(s)\right),t-s\right)\left(\lambda_2(t)^2\lambda_2(s)^4 - \lambda_1(t)^2\lambda_1(s)^4\right)\right]\mathrm{d}s. \end{aligned} \tag{14.2}$$

The numerical method outlined in Section 11 applied to (14.2) leads to a relation between $\lambda_1(s)$ and $\lambda_2(s)$. Let $t = t_1 = 0$. The integral becomes zero and (14.2) reduces to

$$\begin{aligned} 0 = {} & \left(\lambda_2(t_1)^2 - \lambda_1(t_1)^2\right)\left[\alpha_0\left(I\left(C(t_1\right),t_1\right) + \alpha_1\left(I\left(C(t_1\right),t_1\right)\left(\lambda_2(t_1)^2 + \lambda_1(t_1)^2\right)\right. \\ & \left. +\ \alpha_2\left(I\left(C(t_1)\right),t_1\right)\left(\lambda_2(t_1)^4 + \lambda_2(t_1)^2\lambda_1(t_1)^2 + \lambda_1(t_1)^4\right)\right]. \end{aligned} \tag{14.3}$$

It is assumed that the expression in square brackets is not zero. Then the only physically meaningful solution to (14.3) is $\lambda_2(t_1)^2 = \lambda_1(t_1)^2$. Next, evaluate (14.2) at $t = t_2$, introduce the notation $\partial\alpha_i(t-s)/\partial(t-s) = \dot{\alpha}_i(t-s)$ and approximate the integral using the trapezoidal rule as was done to get (11.5) from (11.4). Since $\lambda_2(t_1)^2 = \lambda_1(t_1)^2$, (14.2) reduces to

$$\begin{aligned} 0 = {} & \left(\lambda_2(t_2)^2 - \lambda_1(t_2)^2\right)\left[\alpha_0\left(I\left(C(t_2)\right),t_1\right) + \alpha_1\left(I\left(C(t_2)\right),t_1\right)\left(\lambda_2(t_2)^2 + \lambda_1(t_2)^2\right)\right. \\ & \left. +\ \alpha_2\left(I\left(C(t_2)\right),t_1\right)\left(\lambda_2(t_2)^4 + \lambda_2(t_2)^2\lambda_1(t_2)^2 + \lambda_1(t_2)^4\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}(t_2 - t_1)\left\{[\dot{\alpha}_0(I(C(t_2)), t_1) + \dot{\alpha}_0(I(C(t_1)), t_2 - t_1)]\left(\lambda_2(t_2)^2 - \lambda_1(t_2)^2\right)\right.\\
&+ \left[\dot{\alpha}_1(I(C(t_2)), t_1)\left(\lambda_2(t_2)^2 + \lambda_1(t_2)^2\right) + \dot{\alpha}_1(I(C(t_1)), t_2 - t_1)\lambda_1(t_1)^2\right]\\
&\times \left(\lambda_2(t_2)^2 - \lambda_1(t_2)^2\right) + \left[\dot{\alpha}_2(I(C(t_2)), t_1)\left(\lambda_2(t_2)^4 + \lambda_2(t_2)^2\lambda_1(t_2)^2 + \lambda_1(t_2)^4\right)\right.\\
&\left.\left. + \dot{\alpha}_2(I(C(t_1)), t_2 - t_1)\lambda_1(t_1)^4\right]\left(\lambda_2(t_2)^2 - \lambda_1(t_2)^2\right)\right\}. \qquad (14.4)
\end{aligned}
$$

Since each term in (14.4) has the factor $\lambda_2(t_2)^2 - \lambda_1(t_2)^2$, this equation admits the solution $\lambda_2(t_2)^2 = \lambda_1(t_2)^2$. It is assumed that this is the only physically meaningful solution.

Next, let (14.2) be evaluated at $t = t_n$ and assume that $\lambda_2(t_k)^2 = \lambda_1(t_k)^2, k = 1, 2, \ldots, n-1$. Approximating (14.2) by use of the trapezoidal rule as was done to get (11.7) from (11.6) gives

$$
\begin{aligned}
0 &= \left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right)\left[\alpha_0(I(C(t_n)), t_1) + \alpha_1(I(C(t_n)), t_1)\left(\lambda_2(t_n)^2 + \lambda_1(t_n)^2\right)\right.\\
&\left. + \alpha_2(I(C(t_n)), t_1)\left(\lambda_2(t_n)^4 + \lambda_2(t_n)^2\lambda_1(t_n)^2 + \lambda_1(t_n)^4\right)\right]\\
&+ \sum_{\mathrm{i}=1}^{\mathrm{i}=n-2}\frac{1}{2}(t_{\mathrm{i}+1} - t_{\mathrm{i}})\left\{\left[\dot{\alpha}_0(I(C(t_{\mathrm{i}+1})), t_n - t_{\mathrm{i}+1}) + \dot{\alpha}_0(I(C(t_{\mathrm{i}}))\, t_n - t_{\mathrm{i}})\right]\right.\\
&\times \left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right) + [\dot{\alpha}_1(I(C(t_{\mathrm{i}+1})), t_n - t_{\mathrm{i}+1})\lambda_1(t_{\mathrm{i}+1})^2\\
&+ \dot{\alpha}_1(I(C(t_{\mathrm{i}})), t_n - t_{\mathrm{i}})\lambda_1(t_{\mathrm{i}})^2]\left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right)\\
&+ \left[\dot{\alpha}_2(I(C(t_{\mathrm{i}+1})), t_n - t_{\mathrm{i}+1})\lambda_1(t_{\mathrm{i}+1})^4 + \dot{\alpha}_2(I(C(t_{\mathrm{i}})), t_n - t_{\mathrm{i}})\lambda_1(t_{\mathrm{i}})^4\right]\\
&\times \left.\left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right)\right\} + \frac{1}{2}(t_n - t_{n-1})\{[\dot{\alpha}_0(I(\mathbf{C}(t_n)), t_1)\\
&+ \dot{\alpha}_0(I(\mathbf{C}(t_{n-1})), t_n - t_{n-1})]\left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right) + [\dot{\alpha}_1(I(\mathbf{C}(t_n)), t_1)\\
&\times \left(\lambda_2(t_n)^2 + \lambda_1(t_n)^2\right) + \dot{\alpha}_1(I(\mathbf{C}(t_{n-1})), t_n - t_{n-1})\lambda_1(t_{n-1})^2]\left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right)\\
&+ \left[\dot{\alpha}_2(I(C(t_n)), t_1)\left(\lambda_2(t_n)^4 + \lambda_2(t_n)^2\lambda_1(t_n)^2 + \lambda_1(t_n)^4\right)\right.\\
&\left. + \dot{\alpha}_2(I(C(t_{n-1})), t_n - t_{n-1})\lambda_1(t_{n-1})^4\right]\left(\lambda_2(t_n)^2 - \lambda_1(t_n)^2\right)\}. \qquad (14.5)
\end{aligned}
$$

Since each term in (14.5) has the factor $\lambda_2(t_n)^2 - \lambda_1(t_n)^2$, this equation admits the solution $\lambda_2(t_n)^2 = \lambda_1(t_n)^2$. As before, it is assumed that this is the only physically meaningful solution. This solution holds as t_n increases, that is, as time marches forward. In the limit as the number of time steps increases and the time increments decrease, the approximation to (14.2) is expected to approach the exact equation. Thus, the numerical solution implies that $\lambda_2(s)^2 = \lambda_1(s)^2$, $s \in [0, t]$.

The invariants in (13.6) reduce to

$$
\begin{aligned}
I_1(\mathbf{C}(s)) &= 2\lambda_1(s)^2 + \lambda(s)^2, \\
I_2(\mathbf{C}(s)) &= \lambda_1(s)^4 + 2\lambda_1(s)^2\lambda(s)^2, \\
I_3(\mathbf{C}(s)) &= \lambda_1(s)^4\lambda(s)^2.
\end{aligned}
\tag{14.6}
$$

When the material is compressible, (14.1_1) and (14.1_2), with $p = 0$ give a system of nonlinear Volterra integral equations that relate $\lambda(s)$, $\lambda_1(s)$ and $\sigma(s)$. When the material is incompressible, one finds from the result $\lambda_2(s)^2 = \lambda_1(s)^2$, $s \in [0, t]$ and the condition $\lambda_1(s)\lambda_2(s)\lambda_3(s) = 1$, $s \in [0, t]$ that

$$
\lambda_1(s) = \lambda(s)^{-1/2}, \quad s \in [0, t]. \tag{14.7}
$$

The invariants (14.6) reduce further to

$$
I_1(\mathbf{C}(s)) = \lambda(s)^2 + \frac{2}{\lambda(s)}, \quad I_2(\mathbf{C}(s)) = 2\lambda(s) + \frac{1}{\lambda(s)^2}. \tag{14.8}
$$

The scalar p is found from (14.1_2). Eliminating p between (14.1_1) and (14.1_2) gives

$$
\begin{aligned}
\sigma(t) = {} & \left(\lambda(t)^2 - \frac{1}{\lambda(t)}\right)\left[\alpha_0(I(\mathbf{C}(t)), 0) + \alpha_1(I(\mathbf{C}(t)), 0)\left(\lambda(t)^2 + \frac{1}{\lambda(t)}\right)\right. \\
& + \left.\alpha_2(I(\mathbf{C}(t)), 0)\left(\lambda(t)^4 + \lambda(t) + \frac{1}{\lambda(t)^2}\right)\right] \\
& + \int_0^t \frac{\partial}{\partial(t-s)}\left[\alpha_0(I(\mathbf{C}(s)), t-s)\left(\lambda(t)^2 - \frac{1}{\lambda(t)}\right)\right. \\
& + \alpha_1(I(\mathbf{C}(s)), t-s)\left(\lambda(t)^2\lambda(s)^2 - \frac{1}{\lambda(t)\lambda(s)}\right) \\
& \times \left.\alpha_2(I(\mathbf{C}(s)), t-s)\left(\lambda(t)^2\lambda(s)^4 - \frac{1}{\lambda(t)\lambda(s)^2}\right)\right] \mathrm{d}s,
\end{aligned}
\tag{14.9}
$$

the stress–stretch relation for an isotropic, incompressible nonlinear viscoelastic solid. This equation is the focus of the remainder of this section.

14.2. Small Strain Limit

Equation (14.9) can be expressed in terms of the strain by substituting the relation $\lambda(s) = 1 + \varepsilon(s)$. Assume that $\varepsilon_m = \max|\varepsilon(s)|$, $s \in [0, t]$, is small and expand the right hand side in a Taylor series in $\varepsilon(s)$. The approximation to (14.9) including terms through $o\left(\varepsilon_m^3\right)$ is

$$
\begin{aligned}
\sigma(t) = {} & \varepsilon(t)G(0) + \int_0^t \frac{\mathrm{d}G(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)\mathrm{d}s + \varepsilon(t)\left[\varepsilon(t)G(0) + \int_0^t \frac{\mathrm{d}G(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)\mathrm{d}s\right] \\
& + \varepsilon(t)^2\hat{G}(0) + \int_0^t \frac{\mathrm{d}\hat{G}(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)^2\mathrm{d}s.
\end{aligned}
\tag{14.10}
$$

In deriving (14.10), $\alpha_i(3,3,t)$ denotes $\alpha_i(I(\mathbf{C}),t)$ when the values of the invariants are $I_1(\mathbf{C}) = I_2(\mathbf{C}) = 3$ and use has been made of (8.18). Then

$$G(t) = 3\alpha_1(3,3,t) + 6\alpha_2(3,3,t), \quad \hat{G}(t) = 3\alpha_2(3,3,t), \tag{14.11}$$

Thus, when $\varepsilon_m << 1$ and only first order terms are retained, (14.10) reduces to the constitutive equation for linear viscoelasticity (2.15_1).

14.3. Stress Relaxation

Let $\lambda(s) = \lambda_o \neq 1$, $s \in [0,t]$. By (14.8),

$$I_1(\mathbf{C}(s)) = \lambda_o^2 + 2/\lambda_o, \quad I_2(\mathbf{C}(s)) = 2\lambda_o + 1/\lambda_o^2. \tag{14.12}$$

Introducing the notation, $\tilde{\alpha}_i(\lambda_o, s) = \alpha_i(I(\mathbf{C}(s)), s)$ when the invariants are given by (14.12), (14.9) reduces to

$$\begin{aligned}\sigma(t) &= \tilde{\alpha}_0(\lambda_o,t)\left(\lambda_o^2 - \frac{1}{\lambda_o}\right) + \tilde{\alpha}_1(\lambda_o,t)\left(\lambda_o^4 - \frac{1}{\lambda_o^2}\right) + \tilde{\alpha}_2(\lambda_o,t)\left(\lambda_o^6 - \frac{1}{\lambda_o^3}\right)\\ &= G(t,\lambda_o),\end{aligned} \tag{14.13}$$

a stretch dependent stress relaxation function for uniaxial extension. Setting $\lambda_o = 1 + \varepsilon_o$ in (14.14) gives $G(t, \varepsilon_o)$ introduced in Section 2.2.

Various forms for $G(t,\lambda_o)$ have appeared in the literature:

1. A simple separation of variables product form as in the quasi-linear viscoelastic constitutive equation (6.13),

$$G(t,\lambda_o) = f^{(e)}(\lambda_o)G(t). \tag{14.14}$$

2. A summation of product terms as in the Lianis constitutive equation (12.11),

$$G(t,\lambda_o) = \sum_{k=1}^{N} f_i(\lambda_o)\,\tilde{G}_i(t). \tag{14.15}$$

3. A decomposition such as (2.24_1) with a stretch dependent characteristic time,

$$G(t,\lambda_o) = G_\infty(\lambda_o) + [G_o(\lambda_o) - G_\infty(\lambda_o)]\,\tilde{G}\left[t/\tau(\lambda_o)\right]. \tag{14.16}$$

14.4. Creep

Consider the step stress history $\sigma(t) = \sigma_o$, where σ_o is a constant. Let the creep response to this step stress history be denoted by $J(t,\sigma_o)$, the creep function introduced in Section 2.1.

This is found by solving the nonlinear Volterra integral equation (14.9), the counterpart here of (2.18_2).

Few analytical methods appear to be available for solving nonlinear Volterra integral equation or for gaining information about the solution. When possible, it is usually necessary to assume the form of the nonlinearity. Owing to the general nonlinear dependence of (14.9) on $\lambda(s)$ and the fact that there are many possible forms for the material property functions, it does not seem likely that an analytical solution for $J(t, \sigma_o)$ can be found. Instead, $J(t, \sigma_o)$ will have to be determined numerically using the method described in Section 11.

It is possible, in linear viscoelasticity, to derive analytical relations (2.19) and (2.20) between the creep and stress relaxation properties from (2.18_1) or (2.18_2). It does not seem possible to derive corresponding relations between $J(t, \sigma_o)$ and the material properties in (14.9). It is expected, based on linear viscoelasticity, that if the scalar coefficient $\alpha_i(t)$ monotonically decreases to a non-zero limit as $t \to \infty$, then $J(t, \sigma_o)$ will monotonically increase to a finite limit $J(\infty, \sigma_o)$.

14.5. Isochrones

It has been shown that the nonlinear viscoelastic response given by (14.9) approaches linear viscoelastic response given by (2.15_1) as the magnitude of the maximum strain decreases. This implies that the stress relaxation and creep isochrones approach straight lines through the origin as the maximum strain or stress decreases. This has been demonstrated for both stress relaxation and creep isochrones by Smart and Williams [34].

14.6. Constant Stretch Rate Histories

The stretch history $\lambda(s) = 1 + \alpha s$, $s \in [0, t]$, where α is a constant stretch rate, plays an important role in experimental programs for determining the constitutive equation for nonlinear viscoelastic materials. The stress history calculated from (14.9) can have a complicated dependence on the stretch rate α and time.

Useful insight into the deviation from linear response can be gained by considering stretch rates and initial time intervals when the strain $\varepsilon(s) = \alpha s$ is small. In this case, the stress is found by substituting $\varepsilon(s) = \alpha s$ into (14.10),

$$\sigma(t) = \alpha \int_0^t G(s)\mathrm{d}s + \alpha^2 \left[t \int_0^t G(s)\mathrm{d}s + 2 \int_0^t \hat{G}(s)\,(t - s)\,\mathrm{d}s \right]. \tag{14.17}$$

When $\varepsilon(s) = \alpha s$ is infinitesimal, the stress is given by the first term in (14.17), which is the linear viscoelastic response (2.26). A plot of σ/α vs. t is independent of α. When $\varepsilon(s) = \alpha s$ increases so that the quadratic terms in (14.17) become important, the plot of σ/α vs. t becomes dependent on α. The point at which this occurs indicates the onset of nonlinear response.

Another point of view is obtained by calculating $\mathrm{d}\sigma/\mathrm{d}t$ from (14.17),

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \alpha G(t) + \alpha^2 \left[\int_0^t G(s)\mathrm{d}s + tG(t) + 2 \int_0^t \hat{G}(s)\mathrm{d}s \right]. \tag{14.18}$$

As discussed in Section 2.9, linear viscoelastic response is given by the term in (14.18) that is linear in α. The slope of the plot of σ vs. t plot monotonically decreases in this regime. Now, consider the influence of the α^2 term on the slope. The first term in the square bracket monotonically increases with time. The second term initially increases due to the factor t, but may then decrease due to the factor $G(t)$. The behavior of the third term depends on $\hat{G}(t)$, about which little is known. As the α^2 term becomes important, the slope of the plot of σ vs. t may begin to increase. The point at which the slope deviates from a monotonic decrease is another indication of the onset of nonlinear response.

Constant stress rate histories were discussed for linear viscoelastic response in Section 2.9. Results were developed using the inverse form of the constitutive equation and expressed in terms of the creep compliance. Since (14.9) does not have a simple inverse, the response to a constant stress rate history must be found by solving the nonlinear Volterra integral equation obtained by setting $\sigma(t) = \beta t$ in (14.9).

14.7. Sinusoidal Oscillations About the Reference state

Let the block be subjected to the sinusoidal stretch history $\lambda(s) = 1 + \varepsilon_o \sin \omega s$, $s \in [0, t]$. When this stretch history is substituted into (14.9), it is assumed that the stress reaches a state of steady oscillations. The mathematical issues in showing this are not discussed here. When $|\varepsilon_o| << 1$, the response is linear and is given by (2.31_1). When $|\varepsilon_o|$ is larger, the nonlinear dependence of (14.9) on $\lambda(s)$ causes the σ vs. t plot to be periodic but not sinusoidal. It is interesting to study the change in the σ vs. t plot as $|\varepsilon_o|$ increases.

To this end, let the strain history $\varepsilon(s) = \varepsilon_o \sin \omega s$ be substituted into (14.10). The stress becomes

$$\begin{aligned}\sigma(t) &= \varepsilon_o \left[G'(\omega) \sin \omega t + G''(\omega) \cos \omega t\right] \\ &+ \frac{\varepsilon_o^2}{2} \left[G'(\omega) + \hat{G}(\infty) - \cos 2\omega t \left(G'(\omega) + \hat{G}'(2\omega)\right)\right. \\ &+ \left.\sin 2\omega t \left(G''(\omega) + \hat{G}''(2\omega)\right)\right]. \end{aligned} \tag{14.19}$$

Equation (14.19) shows that as the terms in ε_o^2 become larger, the stress becomes modified by the addition of terms in $\sin 2\omega t$ and $\cos 2\omega t$. Clearly, when $|\varepsilon_o|$ increases further so that terms in $|\varepsilon_o|^n$ must be included, the stress becomes further modified by the addition of terms in $\sin n\omega t$ and $\cos n\omega t$. Thus, an indicator of nonlinearity is the appearance of these higher frequency terms and the associated change of the shape of the σ vs. t plot.

Because of the appearance of these higher frequency terms, stress and strain are no longer related by (2.40). The stress–strain plot changes from an ellipse to some other shape of closed curve. Both squared off and S-shaped curves have been observed in experiments.

14.8. Small Deformation Superposed on Finite Uniaxial Stretch

Let a bar be subjected to a uniaxial stretch history described as follows: a step stretch is applied at $t = 0$ and held constant until the stress has relaxed to its long time value. The stretch then has a small perturbation. The stretch history is given by

$$\lambda(s) = \lambda_o (1 + \eta(s)), \quad s \in [0, t], \tag{14.20}$$

where $|\eta(s)| << 1$ and

$$\begin{aligned} \eta(s) &= 0, \quad s \in [0, T^*] \\ &= \hat{\eta}(s - T^*), \quad s \in [T^*, t]. \end{aligned} \tag{14.21}$$

The stress relaxation response to the underlying step stretch history $\lambda(s) = \lambda_o$, $s \in [0, t]$, is given by (14.13). Let time T^* be large enough that the stress is very close to its long time limit. The response to the perturbed stretch history is obtained by substituting (14.20) and (14.21) into (14.9), expanding terms in Taylor series and retaining only the terms linear in the perturbation $\eta(s)$. Letting $t = \hat{t} + T^*$, where $\hat{t}$ denotes a time measured from T^*, the stress becomes

$$\sigma(t) = G(\infty, \lambda_o) + \tilde{\sigma}(\lambda_o; \hat{t}) \tag{14.22}$$

where

$$\begin{aligned} \tilde{\sigma}(\lambda_o; \hat{t}) = \; & \hat{\eta}(\hat{t}) \left[\left(2\lambda_o^2 + \frac{1}{\lambda_o}\right) \alpha_0(\lambda_o; \infty) + \left(2\lambda_o^4 + \frac{1}{\lambda_o^2}\right) \alpha_1(\lambda_o; \infty) \right. \\ & \left. + \left(2\lambda_o^6 + \frac{1}{\lambda_o^3}\right) \alpha_2(\lambda_o; \infty) \right] + \hat{\eta}(\hat{t}) M(\lambda_o; 0) \\ & + \int_0^{\hat{t}} \hat{\eta}(s) \frac{\partial M(\lambda_o; \hat{t} - s)}{\partial (\hat{t} - s)} \mathrm{d}s, \end{aligned} \tag{14.23}$$

and

$$\begin{aligned} M(\lambda_o; t) = \; & 2\left(\lambda_o^2 - \frac{1}{\lambda_o}\right)^2 \alpha_0^*(\lambda_o; t) \\ & + \left(2\lambda_o^4 + \frac{1}{\lambda_o^2}\right) \alpha_1(\lambda_o; t) + 2\left(\lambda_o^2 - \frac{1}{\lambda_o}\right)\left(\lambda_o^4 - \frac{1}{\lambda_o^2}\right) \alpha_1^*(\lambda_o; t) \\ & + \left(4\lambda_o^6 + \frac{2}{\lambda_o^3}\right) \alpha_2(\lambda_o; t) + 2\left(\lambda_o^2 - \frac{1}{\lambda_o}\right)\left(\lambda_o^6 - \frac{1}{\lambda_o^3}\right) \alpha_2^*(\lambda_o; t), \end{aligned}$$

with

$$\alpha_i^*(\lambda_o;t) = \frac{\partial \alpha_i}{\partial I_1}(\lambda_o;t) + \frac{1}{\lambda_o}\frac{\partial \alpha_i}{\partial I_2}(\lambda_o;t). \tag{14.24}$$

The last two terms in (14.23) have the same form as (2.15_1) and hence represent a superposed linear viscoelastic response with a stress relaxation modulus $M(\lambda_o;t)$ that depends on the underlying stretch λ_o.

Let the perturbation be the sinusoidal deformation

$$\hat{\eta}(s) = \varepsilon_o \sin \omega s. \tag{14.25}$$

When time $\hat{t}$ becomes large, (14.23) becomes

$$\begin{aligned}\tilde{\sigma}(\lambda_o;\hat{t}) =\ & \varepsilon_o \sin\omega\hat{t}\left[\left(2\lambda_o^2 + \frac{1}{\lambda_o}\right)\alpha_0(\lambda_o;\infty) + \left(2\lambda_o^4 + \frac{1}{\lambda_o^2}\right)\alpha_1(\lambda_o;\infty)\right.\\ & \left. + \left(2\lambda_o^6 + \frac{1}{\lambda_o^3}\right)\alpha_2(\lambda_o;\infty)\right] + \varepsilon_o[M'(\lambda_o;\omega)\sin\omega\hat{t}\\ & + M''(\lambda_o;\omega)\cos\omega\hat{t}],\end{aligned} \tag{14.26}$$

with

$$M'(\lambda_o;\omega) = M(\lambda_o;0) + \int_0^\infty \frac{\partial M(\lambda_o;s)}{\partial s}\cos\omega s \mathrm{d}s, \tag{14.27$_1$}$$

$$M''(\lambda_o;\omega) = -\int_0^\infty \frac{\partial M(\lambda_o;s)}{\partial s}\sin\omega s \mathrm{d}s. \tag{14.27$_2$}$$

This result shows the influence of the underlying stretch on the response to superposed small amplitude vibrations. In particular, the underlying stretch affects $M''(\lambda_o;\omega)$. Thus, recalling the discussion in Section 2.10, the underlying stretch affects the damping characteristics and work done per cycle.

Goldberg and Lianis [35] carried out similar calculations using the constitutive equation (8.13) as well as other models, performed experiments involving small amplitude oscillations on finite stretch and compared the results with predictions of the models. Morman et al. [36] and Morman and Nagtegaal [37] extended the ideas illustrated here to general small-amplitude vibrations superposed on large deformations for the Lianis constitutive equation (8.13) and incorporated the results in a finite element analysis.

15. HOMOGENEOUS DEFORMATIONS—BIAXIAL EXTENSION HISTORIES

Biaxial extension is the special case of (13.10) when two of the normal stresses act in a plane and the third normal stress equals zero. If the non-zero stresses act in the $X_1 - X_2$ plane, then $\sigma_{33}(t) = 0$, $t \geq 0$. (13.10) reduces to

$$\sigma_{11}(t) = -p + \alpha_0 (I (\mathbf{C}(t)), 0) \lambda_1(t)^2 + \alpha_1 (I (\mathbf{C}(t)), 0) \lambda_1(t)^4 + \alpha_2 (I (\mathbf{C}(t)), 0) \lambda_1(t)^6$$
$$+ \int_0^t \frac{\partial}{\partial (t-s)} \left[\alpha_0 (I (\mathbf{C}(s)), t-s) \lambda_1(t)^2 + \alpha_1 (I (\mathbf{C}(s)), t-s) \lambda_1(t)^2 \lambda_1(s)^2\right.$$
$$\left. + \alpha_2 (I (\mathbf{C}(s)), t-s) \lambda_1(t)^2 \lambda_1(s)^4\right] \mathrm{d}s, \qquad (15.1_1)$$

$$\sigma_{22}(t) = -p + \alpha_0 (I (\mathbf{C}(t)), 0) \lambda_2(t)^2 + \alpha_1 (I (\mathbf{C}(t)), 0) \lambda_2(t)^4 + \alpha_2 (I (\mathbf{C}(t)), 0) \lambda_2(t)^6$$
$$+ \int_0^t \frac{\partial}{\partial (t-s)} \left[\alpha_0 (I (\mathbf{C}(s)), t-s) \lambda_2(t)^2 + \alpha_1 (I (\mathbf{C}(s)), t-s) \lambda_2(t)^2 \lambda_2(s)^2\right.$$
$$\left. + \alpha_2 (I (\mathbf{C}(s)), t-s) \lambda_2(t)^2 \lambda_2(s)^4\right] \mathrm{d}s, \qquad (15.1_2)$$

$$0 = -p + \alpha_0 (I (\mathbf{C}(t)), 0) \lambda_3(t)^2 + \alpha_1 (I (\mathbf{C}(t)), 0) \lambda_3(t)^4 + \alpha_2 (I (\mathbf{C}(t)), 0) \lambda_3(t)^6$$
$$+ \int_0^t \frac{\partial}{\partial (t-s)} \left[\alpha_0 (I (\mathbf{C}(s)), t-s) \lambda_3(t)^2 + \alpha_1 (I (\mathbf{C}(s)), t-s) \lambda_3(t)^2 \lambda_3(s)^2\right.$$
$$\left. + \alpha_2 (I (\mathbf{C}(s)), t-s) \lambda_3(t)^2 \lambda_3(s)^4\right] \mathrm{d}s. \qquad (15.1_3)$$

It is intended that $(15.1_{1,2,3})$ apply to both compressible materials and incompressible materials in a single expression. When the material is compressible material, $p = 0$ and when it is incompressible, $(15.1_{1,2,3})$ are supplemented by the condition $\lambda_1(s)\lambda_2(s)\lambda_3(s) = 1$, $s \in [0, t]$.

Suppose that the material is compressible. If the histories $\lambda_1(s), \lambda_2(s), s \in [0, t]$ are specified, then (15.1_3) becomes a nonlinear Volterra integral equation for $\lambda_3(s)$, $s \in [0, t]$. Once this is known, $(15.1_{1,2})$ are used to determine $\sigma_{11}(t), \sigma_{22}(t)$, $t > 0$. If the stress histories, $\sigma_{11}(t), \sigma_{22}(t)$, $t > 0$, are specified, $(15.1_{1,2,3})$ becomes a system of nonlinear Volterra integral equations for $\lambda_1(s), \lambda_2(s), \lambda_3(s)$, $s \in [0, t]$.

Suppose that the material is incompressible. Then $\lambda_3(s) = [\lambda_1(s)\lambda_2(s)]^{-1}$, $s \in [0, t]$ and p is found from (15.1_3). The system reduces to

$$\sigma_{\mathrm{ii}}(t) = \alpha_0 (I (\mathbf{C}(t)), 0) \left(\lambda_{\mathrm{i}}(t)^2 - \lambda_3(t)^2\right) + \alpha_1 (I (\mathbf{C}(t)), 0) \left(\lambda_{\mathrm{i}}(t)^4 - \lambda_3(t)^4\right)$$
$$+ \alpha_2 (I (\mathbf{C}(t)), 0) \left(\lambda_{\mathrm{i}}(t)^6 - \lambda_3(t)^6\right)$$
$$+ \int_0^t \frac{\partial}{\partial (t-s)} \left[\alpha_0 (I (\mathbf{C}(s)), t-s) \left(\lambda_{\mathrm{i}}(t)^2 - \lambda_3(t)^2\right)\right.$$
$$+ \alpha_1 (I (\mathbf{C}(s)), t-s) \left(\lambda_{\mathrm{i}}(t)^2 \lambda_{\mathrm{i}}(s)^2 - \lambda_3(t)^2 \lambda_3(s)^2\right)$$
$$\left. + \alpha_2 (I (\mathbf{C}(s)), t-s) \left(\lambda_{\mathrm{i}}(t)^2 \lambda_{\mathrm{i}}(s)^4 - \lambda_3(t)^2 \lambda_3(s)^4\right)\right] \mathrm{d}s, \quad \mathrm{i} = 1, 2. \qquad (15.2)$$

Equation (15.2) represents a system of equations that relate $\lambda_1(t), \lambda_2(t)$ and $\sigma_{11}(t), \sigma_{22}(t)$, $t > 0$. If $\lambda_1(s), \lambda_2(s)$, $s \in [0, t]$ are specified, then (15.2) are used to find the stresses. Otherwise, if any other two histories are specified, (15.2) becomes a system of nonlinear Volterra integral equations for the remaining two histories and can be solved using the numerical method outlined in Section 11.

Biaxial extension histories are used in experiments for determining properties of nonlinear viscoelastic materials. For example, see McGuirt and Lianis [20].

16. HOMOGENEOUS DEFORMATIONS—SIMPLE SHEAR HISTORIES

Simple shear motion occurs when (3.1) has the form

$$\begin{aligned} x_1(s) &= X_1 + K(s)X_2, \\ x_2(s) &= X_2, \quad x_3(s) = X_3, \quad s \in [0, t]. \end{aligned} \tag{16.1}$$

The deformation gradient history is

$$\begin{aligned} \mathbf{F}(s) &= \mathbf{I}, \quad s \in (-\infty, 0) \\ \mathbf{F}(s) &= \begin{bmatrix} 1 & K(s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s \in [0, t]. \end{aligned} \tag{16.2}$$

The history of the relative deformation gradient is

$$\mathbf{F}_t(s) = \begin{bmatrix} 1 & -K(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s \in (-\infty, 0) \tag{16.3$_1$}$$

$$\mathbf{F}_t(s) = \begin{bmatrix} 1 & K(s) - K(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s \in [0, t]. \tag{16.3$_2$}$$

Since $\det \mathbf{F}(s) = 1$, $s \in (-\infty, t]$, this motion can be produced in both compressible and incompressible materials.

The kinematic quantities used to calculate stresses from the finite linear viscoelastic and Pipkin–Rogers constitutive equations are:

$$\mathbf{C}(s) = \begin{bmatrix} 1 & K(s) & 0 \\ K(s) & 1 + K(s)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s \in [0, t], \tag{16.4}$$

$$\mathbf{B}(t) = \begin{bmatrix} 1+K(t)^2 & K(t) & 0 \\ K(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{16.5}$$

$$\mathbf{C}_t(s) = \begin{bmatrix} 1 & -K(t) & 0 \\ -K(t) & 1+K(t)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s \in (-\infty, 0) \tag{16.6$_1$}$$

$$\mathbf{C}_t(s) = \begin{bmatrix} 1 & K(s)-K(t) & 0 \\ K(s)-K(t) & 1+(K(s)-K(t))^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s \in [0, t] \tag{16.6$_2$}$$

$$\begin{aligned} I_1\left(\mathbf{B}(t)\right) &= I_2\left(\mathbf{B}(t)\right) = 3 + K(t)^2, \\ I_1\left(\mathbf{C}(s)\right) &= I_2\left(\mathbf{C}(s)\right) = 3 + K(s)^2, \\ I_3\left(\mathbf{C}(s)\right) &= I_3\left(\mathbf{B}(t)\right) = 1. \end{aligned} \tag{16.7}$$

Space limitations permit the stress components for only the Pipkin–Rogers constitutive equation to be presented here. It is straightforward to find them finite for linear viscoelasticity.

$$\begin{aligned} \sigma_{11}(t) &= -p + \alpha_0\left(I\left(\mathbf{C}(t)\right), 0\right)\left(1 + K(t)^2\right) + \alpha_1\left(I\left(\mathbf{C}(t)\right), 0\right)\left(1 + 3K(t)^2 + K(t)^4\right) \\ &+ \alpha_2\left(I\left(\mathbf{C}(t)\right), 0\right)\left(1 + 6K(t)^2 + 5K(t)^4 + K(t)^6\right) \\ &+ \left(1 + K(t)^2\right)\int_0^t \frac{\partial\alpha_0\left(I\left(\mathbf{C}(s)\right), t-s\right)}{\partial\left(t-s\right)}\mathrm{d}s \\ &+ \int_0^t \frac{\partial\alpha_1\left(I\left(\mathbf{C}(s)\right), t-s\right)}{\partial\left(t-s\right)}\left(1 + K(t)^2 + 2K(t)K(s) + K(t)^2K(s)^2\right)\mathrm{d}s \\ &+ \int_0^t \frac{\partial\alpha_2\left(I\left(\mathbf{C}(s)\right), t-s\right)}{\partial\left(t-s\right)}\left(1 + K(t)^2 + K(s)^2 + 4K(t)K(s)\right. \\ &+ \left.2K(t)K(s)^3 + 3K(t)^2K(s)^2 + K(t)^2K(s)^4\right)\mathrm{d}s, \end{aligned} \tag{16.8$_1$}$$

$$\begin{aligned} \sigma_{22}(t) &= -p + \alpha_0\left(I\left(\mathbf{C}(t)\right), 0\right) + \alpha_1\left(I\left(\mathbf{C}(t)\right), 0\right)\left(1 + K(t)^2\right) \\ &+ \alpha_2\left(I\left(\mathbf{C}(t)\right), 0\right)\left(1 + 3K(t)^2 + K(t)^4\right) + \int_0^t \frac{\partial\alpha_0\left(I\left(\mathbf{C}(s)\right), t-s\right)}{\partial\left(t-s\right)}\mathrm{d}s \\ &+ \int_0^t \frac{\partial\alpha_1\left(I\left(\mathbf{C}(s)\right), t-s\right)}{\partial\left(t-s\right)}\left(1 + K(s)^2\right)\mathrm{d}s \\ &+ \int_0^t \frac{\partial\alpha_2\left(I\left(\mathbf{C}(s)\right), t-s\right)}{\partial\left(t-s\right)}\left(1 + 3K(s)^2 + K(s)^4\right)\mathrm{d}s, \end{aligned} \tag{16.8$_2$}$$

$$\begin{aligned}
\sigma_{33}(t) = & -p + \alpha_0 \left(I \left(\mathbf{C}(t) \right), 0 \right) + \alpha_1 \left(I \left(\mathbf{C}(t) \right), 0 \right) + \alpha_2 \left(I \left(\mathbf{C}(t) \right), 0 \right) \\
& + \int_0^t \frac{\partial}{\partial (t-s)} (\alpha_0 \left(I \left(\mathbf{C}(s) \right), t-s \right) + \alpha_1 \left(I \left(\mathbf{C}(s) \right), t-s \right) \\
& + \alpha_2 \left(I \left(\mathbf{C}(s) \right), t-s \right)) \mathrm{d}s, \qquad (16.8_3)
\end{aligned}$$

$$\begin{aligned}
\sigma_{12}(t) = & \ \alpha_0 \left(I \left(\mathbf{C}(t) \right), 0 \right) K(t) + \alpha_1 \left(I \left(\mathbf{C}(t) \right), 0 \right) K(t) \left(2 + K(t)^2 \right) \\
& + \alpha_2 \left(I \left(\mathbf{C}(t) \right), 0 \right) K(t) \left(3 + 4K(t)^2 + K(t)^4 \right) \\
& + K(t) \int_0^t \frac{\partial \alpha_0 \left(I \left(\mathbf{C}(s) \right), t-s \right)}{\partial (t-s)} \mathrm{d}s + \int_0^t \frac{\partial \alpha_1 \left(I \left(\mathbf{C}(s) \right), t-s \right)}{\partial (t-s)} (K(t) + K(s) \\
& + K(t) K(s)^2) \mathrm{d}s + \int_0^t \frac{\partial \alpha_2 \left(I \left(\mathbf{C}(s) \right), t-s \right)}{\partial (t-s)} (2K(s) + K(s)^3 + K(t) \\
& + 3K(t)K(s)^2 + K(t)K(s)^4) \mathrm{d}s, \qquad (16.8_4)
\end{aligned}$$

$$\sigma_{13}(t) = \sigma_{23}(t) = 0. \qquad (16.8_5)$$

For the sake of brevity, (16.8_{1-3}) apply to both compressible materials and incompressible materials in a single expression, with $p = 0$ when the material is compressible.

Equation (16.8_4), with (16.7), relates the shear stress history to the shear history. They show that the shear stress is odd in the shear history, i.e. if $K(s)$ is replaced by $-K(s)$, $s \in [0, t]$, then $\sigma_{12}(t)$ is replaced by $-\sigma_{12}(t)$, $t \geq 0$. Equations (16.8_{1-3}) show that normal stresses are required to produce a simple shearing motion. With Equations (16.7) and (16.8_4), the analyses and results presented in Sections 14.2–14.8 for uniaxial extension are readily extended to shear response.

An interesting result for isotropic nonlinear elastic solids is the relation

$$\sigma_{11} - \sigma_{22} = K \sigma_{12}, \qquad (16.9)$$

discovered by Rivlin [38]. This relation, being independent of material properties and therefore valid for all isotropic elastic solids, is called a *universal relation* for isotropic elastic solids. Using the relations (16.8_{1-4}), one finds that the expression $\sigma_{11}(t) - \sigma_{22}(t) - K(t)\sigma_{12}(t)$ does not vanish because of the presence of integral terms in the constitutive equation. The development of additional universal relations in nonlinear elasticity has received a great deal of attention. It does not seem possible to extend these to nonlinear viscoelasticity. The concept of a universal relation appears to be one associated only with theories in which the constitutive equations do not consider the history of the motion.

17. NON-HOMOGENEOUS DEFORMATIONS

Carroll [39, 40] and Fosdick [41] have shown that there are five families of non-homogeneous motions that are possible in any incompressible isotropic solid, whatever the form of the constitutive equation. Although the motions are independent of the form of the constitutive equation, the stresses are not. For each family of motions, Carroll [39, 40] and Fosdick [41] showed that the stresses are such that an expression for the scalar p can be developed so as to satisfy the equations of motion. These families of motions have been termed *controllable*.

The families of motions are listed below, each with its deformation gradient. The relevant kinematical quantities can then be calculated and used in any constitutive equation of interest. Expressions for the stresses and the scalar p for a particular constitutive equation are not provided in this section. However, in the next section, an example is presented for family (3) using the Pipkin–Rogers constitutive equation.

1. *Bending, stretching and shearing of a rectangular block.*
   The block is described with respect to Cartesian coordinates (X, Y, Z) in its reference configuration and with respect to cylindrical coordinates (r, θ, z) when $t > 0$. The motion (3.1) is described by

$$\begin{aligned} r(s) &= \sqrt{2A(s)(X + D(s))}, \\ \theta(s) &= B(s)(Y + E(s)), \\ z(s) &= \frac{Z}{A(s)B(s)} - B(s)C(s)Y + F(s), \quad s \in [0, t] \end{aligned} \tag{17.1}$$

   where $A(s)B(s) \neq 0$. The deformation gradient is given by

$$\mathbf{F}(s) = \begin{bmatrix} \dfrac{A(s)}{r(s)} & 0 & 0 \\ 0 & r(s)B(s) & 0 \\ 0 & -B(s)C(s) & \dfrac{1}{A(s)B(s)} \end{bmatrix}. \tag{17.2}$$

2. *Straightening, stretching and shearing of a sector of a hollow cylinder.*
   The sector is described with respect to cylindrical coordinates (R, Θ, Z) in its reference configuration and with respect to Cartesian coordinates (x, y, z) when $t > 0$. The motion (3.1) is given by

$$\begin{aligned} x(s) &= \frac{A(s)B(s)^2}{2}R^2 + D(s), \\ y(s) &= \frac{\Theta}{A(s)B(s)} + E(s), \\ z(s) &= \frac{Z}{B(s)} + \frac{C(s)\,\Theta}{A(s)B(s)} + F(s), \quad s \in [0, t], \end{aligned} \tag{17.3}$$

where $A(s)B(s) \neq 0$. The deformation gradient is given by

$$\mathbf{F}(s) = \begin{bmatrix} A(s)B(s)^2R & 0 & 0 \\ 0 & \dfrac{1}{RA(s)B(s)} & 0 \\ 0 & \dfrac{C(s)}{RA(s)B(s)} & \dfrac{1}{B(s)} \end{bmatrix}. \tag{17.4}$$

3. *Inflation, torsion, extension and shearing of an annular wedge.*
   The body is described with respect to cylindrical coordinates (R, Θ, Z) in its reference configuration and with respect to cylindrical coordinates (r, θ, z) when $t > 0$. The motion (3.1) is given by

$$\begin{aligned} r(s) &= \sqrt{A(s)R^2 + B(s)}, \\ \theta(s) &= C(s)\Theta + D(s)Z + G(s), \\ z(s) &= E(s)\Theta + F(s)Z + H(s), \end{aligned} \tag{17.5}$$

   where $A(s)\,(C(s)F(s) - D(s)E(s)) = 1$. The deformation gradient is given by

$$\mathbf{F}(s) = \begin{bmatrix} \dfrac{A(s)R}{r(s)} & 0 & 0 \\ 0 & \dfrac{C(s)r(s)}{R} & r(s)D(s) \\ 0 & \dfrac{E(s)}{R} & F(s) \end{bmatrix}. \tag{17.6}$$

4. *Inflation of a sector of a spherical shell.*
   The body is described with respect to spherical coordinates (R, Θ, Φ) in its reference configuration and with respect to spherical coordinates (r, θ, ϕ) when $t > 0$. The motion (3.1) is given by

$$r(s) = (R^3 + A(s))^{1/3}, \quad \theta = \Theta, \quad \phi = \Phi. \tag{17.7}$$

   The deformation gradient is given by

$$\mathbf{F}(s) = \mathrm{diag}\left[\left(\frac{R}{r(s)}\right)^2, \frac{r(s)}{R}, \frac{r(s)}{R}\right]. \tag{17.8}$$

5. *Inflation, bending, extension and azimuthal shearing of an annular wedge.*
   The body is described with respect to cylindrical coordinates (R, Θ, Z) in its reference configuration and with respect to cylindrical coordinates (r, θ, z) when $t > 0$. The motion (3.1) is given by

$$
\begin{aligned}
r(s) &= A(s)R,\\
\theta(s) &= B(s)\log R + C(s)\Theta + D(s),\\
z(s) &= E(s)Z + F(s),
\end{aligned}
\tag{17.9}
$$

$A(s)^2C(s)E(s) = 1$. The deformation gradient is given by

$$
\mathbf{F}(s) = \begin{bmatrix} A(s) & 0 & 0 \\ A(s)B(s) & A(s)C(s) & 0 \\ 0 & 0 & E(s) \end{bmatrix}. \tag{17.10}
$$

The conditions on the time dependent parameters in each motion arise from imposing the constraint (5.10). There are also conditions on these parameters that ensure that an acceleration **a** can be derived so that (4.3) is satisfied. These conditions and their theoretical support can also be found in the article by Truesdell and Noll [13].

18. NON-HOMOGENEOUS DEFORMATIONS—FURTHER DISCUSSION OF FAMILY (3)

Motion (17.5) contains two important special cases, the combined extension, inflation and torsion of a hollow circular cylinder and the combined tension and torsion of a solid cylinder. Carroll [40] discussed these cases for a general isotropic incompressible solid. The latter motion was used by Yuan and Lianis [42] as part of an experimental program to develop the form of the constitutive equation for finite linear viscoelasticity in (8.13). An example is presented here of the combined tension and torsion of a solid cylinder using the Pipkin–Rogers constitutive equation (8.16).

In its reference configuration the radius of the cylinder is R_o and its length is L_o. At time $s \in [0, t]$, the radius is $r_o(s)$ and its length is $L(s)$. Axial forces $N(s)$ and twisting moments $M(s)$ are applied to the end faces of the cylinder and its curved surface is traction free. The motion is the special case of (17.7),

$$
\begin{aligned}
r(s)^2 &= \frac{R^2}{\lambda(s)},\\
\theta(s) &= \Theta + \psi(s)\lambda(s)Z,\\
z(s) &= \lambda(s)Z.
\end{aligned}
\tag{18.1}
$$

The deformation gradient (17.6) reduces to

$$\mathbf{F}(s) = \begin{bmatrix} 1/\sqrt{\lambda(s)} & 0 & 0 \\ 0 & 1/\sqrt{\lambda(s)} & r(s)\psi(s)\lambda(s) \\ 0 & 0 & \lambda(s) \end{bmatrix}, \tag{18.2}$$

from which important quantities for use in the constitutive equation are found to be

$$\mathbf{C}(s) = \begin{bmatrix} 1/\lambda(s) & 0 & 0 \\ 0 & 1/\lambda(s) & r(s)\psi(s)\sqrt{\lambda(s)} \\ 0 & r(s)\psi(s)\sqrt{\lambda(s)} & \lambda(s)^2 + r(s)^2\psi(s)^2\lambda(s)^2 \end{bmatrix},$$

$$\begin{aligned} I_1 &= 2/\lambda(s) + \lambda(s)^2 + r(s)^2\psi(s)^2\lambda(s)^2, \\ I_2 &= 1/\lambda(s)^2 + 2\lambda(s) + r(s)^2\psi(s)^2\lambda(s). \end{aligned} \tag{18.3}$$

Note that $\mathbf{C}(s)$, I_1 and I_2 can be expressed in terms of either $r(s)$ or R by use of (18.1_1).

The components of $\mathbf{\Pi}(t)$ in (8.17) can be calculated using (18.3). The calculation is straightforward and the resulting expressions are omitted for the purpose of brevity. Let the stress be denoted by

$$\boldsymbol{\sigma}(t) = -p\mathbf{I} + \mathcal{F}(R, t) = -p\mathbf{I} + \bar{\mathcal{F}}(r(t), t), \tag{18.4}$$

where (18.1_1) is used to change the independent variable from R to $r(s)$. By (8.16) and (18.2), it is found that

$$\begin{aligned} \mathcal{F}_{rr}(R, t) &= \Pi_{rr}/\lambda, \\ \mathcal{F}_{\theta\theta}(R, t) &= \Pi_{\theta\theta}/\lambda + 2\psi R\Pi_{\theta z} + \lambda\psi^2 R^2\Pi_{zz}, \\ \mathcal{F}_{zz}(R, t) &= \lambda^2\Pi_{zz}, \\ \mathcal{F}_{\theta z}(R, t) &= \sqrt{\lambda}\Pi_{\theta z} + \lambda\sqrt{\lambda}R\psi\Pi_{zz}, \\ \mathcal{F}_{r\theta}(R, t) &= F_{rz}(R, t) = 0. \end{aligned} \tag{18.5}$$

According to (18.4) and (18.5), there can only be a normal traction on the outer surface of the cylinder. As this surface is free of traction,

$$\sigma_{rr}(r_o(t), t) = 0. \tag{18.6}$$

This motion and the stresses given by (18.4) and (18.5) must satisfy (4.3) at time t and for $r \in [0, r_o(t)]$. Both the body force and inertia are neglected here. As shown in the article by Truesdell and Noll [13], these can be included in a straightforward manner if the body force can be derived from a potential. Since $\sigma_{r\theta} = \sigma_{rz} = 0$ and $\bar{\mathcal{F}}$ depends only on the current radius, (4.3) implies that $p = p(r(t), t)$ and the remaining stresses must satisfy

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{18.7}$$

By use of (18.4)–(18.6), an expression can be found for the scalar "p" that ensures satisfaction of (18.7). This expression for "p" and the resultant expressions for the stresses obtained from (18.4) are given by

$$-p(t) = -\bar{\mathcal{F}}_{rr} + \int_r^{r_o(t)} \frac{1}{\bar{r}} \left[\bar{\mathcal{F}}_{\theta\theta} - \bar{\mathcal{F}}_{rr}\right] \mathrm{d}\bar{r}, \tag{18.8$_1$}$$

$$\sigma_{rr}(t) = -\int_r^{r_o(t)} \frac{1}{\bar{r}} \left[\bar{\mathcal{F}}_{\theta\theta} - \bar{\mathcal{F}}_{rr}\right] \mathrm{d}\bar{r}, \tag{18.8$_2$}$$

$$\sigma_{\theta\theta} = \bar{\mathcal{F}}_{\theta\theta} - \bar{\mathcal{F}}_{rr} - \int_r^{r_o(t)} \frac{1}{\bar{r}} \left[\bar{\mathcal{F}}_{\theta\theta} - \bar{\mathcal{F}}_{rr}\right] \mathrm{d}\bar{r}, \tag{18.8$_3$}$$

$$\sigma_{zz} = \bar{\mathcal{F}}_{zz} - \bar{\mathcal{F}}_{rr} - \int_r^{r_o(t)} \frac{1}{\bar{r}} \left[\bar{\mathcal{F}}_{\theta\theta} - \bar{\mathcal{F}}_{rr}\right] \mathrm{d}\bar{r}. \tag{18.8$_4$}$$

The axial force and twisting moment on the ends of the cylinder are the resultants of stresses given by

$$N(t) = 2\pi \int_0^{r_o(t)} \sigma_{zz}\left(r(t), t\right) r \mathrm{d}r. \tag{18.9$_1$}$$

$$M(t) = 2\pi \int_0^{r_o(t)} r\sigma_{\theta z}\left(r, t\right) r \mathrm{d}r. \tag{18.9$_2$}$$

The integrals in ($18.9_{1,2}$) are defined over the current configuration. They can be transformed into integrals over the reference configutration by evaluating (18.1_1) at $s = t$ to give $r(t) = R/\sqrt{\lambda(t)}$, a relation between a radius in the configuration at time t and its corresponding radius in the reference configuration. On substituting (18.8_4) into (18.9_1), integrating by parts to simplify the expression and then changing variables, the expression for the axial force becomes

$$N(t) = 2\pi \int_0^{R_o} \left[2\left(\mathcal{F}_{zz} - \mathcal{F}_{rr}\right) - \left(\mathcal{F}_{\theta\theta} - \mathcal{F}_{rr}\right)\right] R \mathrm{d}R. \tag{18.10}$$

In a similar manner, the expression for the twisting moment becomes

$$M(t) = 2\pi \int_0^{R_o} \frac{1}{\lambda\sqrt{\lambda}} \mathcal{F}_{\theta z} R^2 \mathrm{d}R. \tag{18.11}$$

Let expressions for $F_{\theta z}$, $F_{zz} - F_{rr}$ and $F_{\theta\theta} - F_{rr}$ be calculated from (8.17) and (18.5) and then substituted into (18.10) and (18.11). The axial force and twisting moment are then given by

$$M(t) = M'(\lambda, \psi) + \int_0^t M''(\lambda, \psi, \lambda(s), \psi(s), t-s)\,\mathrm{d}s, \tag{18.12}$$

$$N(t) = N'(\lambda, \psi) + \int_0^t N''(\lambda, \psi, \lambda(s), \psi(s), t-s)\,\mathrm{d}s, \tag{18.13}$$

in which

$$\begin{aligned} M'(\lambda, \psi) = {} & 2\pi \int_0^{R_o} R^2 \frac{1}{\lambda} \{\alpha_0 (I(\mathbf{C}(t)), 0)\, R\psi\lambda \\ & + \alpha_1 (I(\mathbf{C}(t)), 0) \left[R\psi + R\psi\lambda \left(\lambda^2 + R^2\psi^2\lambda\right)\right] \\ & + \alpha_2 (I(\mathbf{C}(t)), 0) \left[R\psi \left(\frac{1}{\lambda} + \lambda^2 + R^2\psi^2\lambda\right)\right. \\ & \left.\left. + R\psi\lambda \left(R^2\psi^2 + \left(\lambda^2 + R^2\psi^2\lambda\right)^2\right)\right]\right\} \mathrm{d}R \end{aligned} \tag{18.14$_1$}$$

$$\begin{aligned} & M''(\lambda, \psi, \lambda(s), \psi(s), t-s) \\ & = 2\pi \int_0^{R_o} R^2 \frac{1}{\lambda} \left\{ \frac{\partial\alpha_0 (I(\mathbf{C}(s)), t-s)}{\partial(t-s)} R\psi\lambda \right. \\ & + \frac{\partial\alpha_1 (I(\mathbf{C}(s)), t-s)}{\partial(t-s)} \left[R\psi(s) + R\psi\lambda \left(\lambda(s)^2 + R^2\psi(s)^2\lambda(s)\right)\right] \\ & + \frac{\partial\alpha_2 (I(\mathbf{C}(s)), t-s)}{\partial(t-s)} \left[R\psi(s) \left(\frac{1}{\lambda(s)} + \lambda(s)^2 + R^2\psi(s)^2\lambda(s)\right)\right. \\ & \left. + R\psi\lambda \left(R^2\psi(s)^2 + (\lambda(s))^2 + R^2\psi(s)^2\lambda(s)\right)^2\right], \end{aligned} \tag{18.14$_2$}$$

$$\begin{aligned} N'(\lambda, \psi) = {} & 2\pi \int_0^{R_o} \frac{1}{2\lambda} R \{\alpha_0 [I(\mathbf{C}(t)), 0] \left[2\left(\lambda^2 - \frac{1}{\lambda}\right) - R^2\psi^2\lambda\right] \\ & + \alpha_1 [I(\mathbf{C}(t)), 0] \left[2\left(\lambda^4 - \frac{1}{\lambda^2}\right) + R^2\psi^2\lambda^3 - 2R^2\psi^2 - R^4\psi^4\lambda^2\right] \\ & + \alpha_2 [I(\mathbf{C}(t)), 0] \left[\left(2\lambda^2 - R^2\psi^2\lambda\right) \left(R^2\psi^2 + \left(\lambda^2 + R^2\psi^2\lambda\right)^2\right)\right. \\ & \left.\left. - \frac{3}{\lambda} R^2\psi^2 - 2R^2\psi^2 \left(\lambda^2 + R^2\psi^2\lambda\right)\right]\right\} \mathrm{d}R, \end{aligned} \tag{18.15$_1$}$$

$$\begin{aligned} & N''(\lambda, \psi, \lambda(s), \psi(s), t-s) \\ & = 2\pi \int_0^{R_o} \frac{1}{2\lambda} R \left\{ \frac{\partial\alpha_0 (I(\mathbf{C}(s)), t-s)}{\partial(t-s)} \left(2\left(\lambda^2 - \frac{1}{\lambda}\right) - R^2\psi^2\lambda\right) \right. \end{aligned}$$

$$+ \frac{\partial \alpha_1 (I(\mathbf{C}(s)), t-s)}{\partial (t-s)} \Big[\left(\lambda(s)^2 + R^2 \psi(s)^2 \lambda(s)\right) \left(2\lambda^2 - R^2 \psi^2 \lambda\right)$$

$$- \frac{2}{\lambda \lambda(s)} - 2R^2 \psi \psi(s) \Big] + \frac{\partial \alpha_2 (I(\mathbf{C}(s)), t-s)}{\partial (t-s)}$$

$$\times \Big[\left(R^2 \psi(s)^2 + \left(\lambda(s)^2 + R^2 \psi(s)^2 \lambda(s)\right)^2\right) \left(2\lambda^2 - R^2 \psi^2 \lambda\right) - \frac{2}{\lambda \lambda(s)^2}$$

$$- \frac{R^2 \psi(s)^2}{\lambda} - 2R^2 \psi \psi(s) \left(\frac{1}{\lambda(s)} + \lambda(s)^2 + R^2 \psi(s)^2 \lambda(s) \right) \Big] \Big\} \, \mathrm{d}R. \qquad (18.15_2)$$

If either $\lambda(s)$ or $\psi(s)$, $s \in [0, t]$, is specified, then $N(t)$ and $M(t)$ are determined from (18.12)–(18.15). On the other hand, suppose $N(s)$ and $M(s)$, $s \in [0, t]$, are specified. Although ($18.14_{1,2}$) and ($18.15_{1,2}$) involve definite integrals over R, they define functions of their indicated arguments. Thus, (18.12) and (18.13) become a system of coupled nonlinear Volterra integral equation of form (11.1) for $\lambda(s)$ and $\psi(s)$ and can be solved by the method described in Section 11.

19. SOLUTION OF BOUNDARY VALUE PROBLEMS

Nonlinearity in viscoelastic response occurs when there is large deformation and/or nonlinear material properties. The following describes a variety of solutions to nonlinear viscoelastic boundary value problems that have appeared in the literature. It is not intended that an exhaustive summary of such solutions be provided here. The purpose is to provide a representative listing of solutions to nonlinear viscoelasticity problems that have been obtained and to thereby show that such solutions are quite feasible.

19.1. Finite Deflection of Viscoelastic Beams

Lee and Rogers [43] described the finite deflection of a viscoelastic cantilever beam under a time dependent concentrated force at its free end. In its undeformed state, the beam thickness was assumed to be small compared to its length. Points on the neutral axis were allowed to undergo large displacement while the strains through the thickness were assumed to be small. With reference to (5.5), the rotations $\mathbf{R}(t)$ were large, $\mathbf{U}(t-s)$ was approximated by $\mathbf{U}(t-s) \approx \mathbf{I} + \mathbf{e}(t-s)$, where $\mathbf{e}(t-s)$ is the small strain tensor and $\mathcal{F}$ was approximated by the constitutive equation (2.41) for linearized viscoelasticity. This allowed use of the assumptions of classical beam theory.

The neutral axis was assumed to be inextensible and of length L. It coincided with the X_1-axis in the undeformed state so that each point was labeled by its initial coordinate X_1, $0 \leq X_1 \leq L$. Let $\phi(X_1, t)$ denote the angle between the tangent to the point on the deformed neutral axis at X_1 and the x_1-axis at time t. The formulation accounted for the application of the concentrated force in the current configuration and led to the partial differential-Volterra integral equation for $\phi(X_1, t)$,

$$\frac{\partial^2 \phi(X_1, t)}{\partial X_1^2} = -\frac{L^2 J(0)}{I}\left[P(t)\cos\phi(X_1, t) + \int_0^t \frac{1}{J(0)}\frac{\mathrm{d}J(t-s)}{\mathrm{d}(t-s)} P(s)\cos\phi(X_1, s)\,\mathrm{d}s\right], \tag{19.1}$$

in which I is the second area moment of the cross section, $P(t)$ is the applied force at time t and $J(t)$ is the creep compliance in uniaxial extension. A numerical method of solution was discussed that extended the approach presented in Section 11, and results were presented that showed the deflection history of the neutral axis.

19.2. Nonlinear Viscoelastic Membranes

A number of solutions to problems involving large deformations of viscoelastic membranes have appeared in the literature. For the examples mentioned here, the material is incompressible, isotropic and is described by one of nonlinear single integral constitutive equations described in Section 8, that is, either the Lianis constitutive equation for finite linear viscoelasticity (8.13), the Pipkin–Rogers constitutive equation (8.16) and (8.17) or the K-BKZ constitutive equation (8.19).

Several problems [44, 45] were solved for large in-plane radial axisymmetric deformations of an initially plane annular membrane. The Pipkin–Rogers constitutive equation (8.16) was used, with material parameters being chosen to combine features of the Mooney–Rivlin model of nonlinear elasticity with the three parameter solid of linear viscoelasticity. Boundary conditions were applied at the inner and outer boundary. In [44], the inner boundary was traction free. Two cases were considered, one with tractions and one with displacement specified at the outer boundary. In [45], the inner boundary was fixed, the outer boundary was traction free and the membrane deformed due to centrifugal force while spinning. The formulation led to a system of equations of the form

$$\begin{aligned}
&\hat{D}\left(\lambda_{\mathrm{i}}(t), \frac{\partial \lambda_{\mathrm{i}}(t)}{\partial R}\right)\left\{F_{1}(\Lambda(t)) + \int_0^t G_{1\mathrm{i}}(\Lambda(t), \Lambda(s), t-s)\,\mathrm{d}s\right\} \\
&+ \int_0^t \hat{\hat{D}}\left(\lambda_{\mathrm{i}}(s), \frac{\partial \lambda_{\mathrm{i}}(s)}{\partial R}\right) G_{2\mathrm{i}}(\Lambda(t), \Lambda(s), t-s)\,\mathrm{d}s \\
&= F_{2\mathrm{i}}(\Lambda(t), r) + \int_0^t G_{3\mathrm{i}}(\Lambda(t), \Lambda(s), t-s, r)\,\mathrm{d}s
\end{aligned} \tag{19.2}$$

in which $\Lambda(t) = (\lambda_1(t), \lambda_2(t))$, λ_1 being the stretch ratio in the radial direction and λ_2 being the stretch ratio in the circumferential direction. $\hat{D}$ and $\hat{\hat{D}}$, being functions of λ_{i} and its partial derivative with respect to R, define partial differential operators on λ_{i}. Thus, these were two point boundary value problems involving the nonlinear partial differential-Volterra integral equation (19.2).

Several problems have been solved for out-of–plane deformations of initially plane membranes clamped along a circular boundary. In [46] and [47], the membrane was inflated to a surface of revolution by lateral pressure. In [48], the membrane was deformed axisymmetrically by a spherical indenter. The Lianis model was used in [46], a special case of the Pipkin–Rogers model was used in [47] and the K-BKZ model was used in [48]. In another application [49], a tubular membrane was attached to rigid discs at its ends. The membrane was deformed by internal pressure and axial forces were applied to the discs. In this case the material was taken to be a BKZ fluid. Each formulation led to a two-point boundary value problem with equations similar to (19.2), but with $\mathbf{\Lambda}(t) = (\lambda_1(t), \lambda_2(t), \eta(t))$, $\eta(t)$ being an associated kinematical variable. In each problem, a method of numerical solution was developed that combined the approach outlined in Section 11 with that of Lee and Rogers [43]. The inflation of a spherical membrane by internal pressure in [50] showed an interesting phenomenon due to combined large deformation and viscoelasticity. There can be a time when the solution develops branches. A general discussion of axisymmetric problems involving nonlinear viscoelastic membranes is given in [51].

19.3. Shearing of Viscoelastic Cylinders

Elastomers are rubbery nonlinear viscoelastic materials that are used in a variety of components in vehicles. One class of components, used as part of vehicle suspension systems, consists of bushings. A bushing is essentially a hollow cylinder of elastomeric material contained between an inner metal rod and an outer metal sleeve. The sleeve and rod are connected to components of the suspension system. They undergo relative motions along and about their common centerline as well as along and about axes perpendicular to the centerline. The relation between the forces and moments applied to the rod and sleeve and their relative motion is used in the engineering of suspension systems. Models have recently been developed that relate forces and relative displacements along the common centerline (axial mode) [52], moments and relative rotations about the common centerline (torsional mode) [53] and their coupled response [54]. The constitutive equation for the bushing material was taken to be that of Lianis in (8.13). The cylinder was assumed to undergo axisymmetric linear displacements $w(R,t)$ along the centerline and/or rotational displacements $g(R,t)$ about the centerline, both varying with the radius. As a result, each material element experienced an axial shear $k(R,t) = \partial w(R,t)/\partial R$ and a circumferential shear $h(R,t) = R\partial g(R,t)/\partial R$. The equations of motion, with cylinder inertia neglected, led to relations between the shear stress σ_{rz} and axial force $F(t)$, $\sigma_{rz}(R,t) = F(t)/2\pi RL$ and between the shear stress $\sigma_{r\theta}$ and moment about the centerline $M(t)$, $\sigma_{r\theta}(R,t) = M(t)/2\pi R^2 L$. The formulation reduced to an equation of form (11.1) with $\mathbf{f}(R,t) = \left[\sigma_{rz}(R,t), \sigma_{r\theta}(R,t)\right]$ and $\mathbf{x}(R,t) = [k(R,t), h(R,t)]$. This provides a connection between the histories of axial force $F(t)$ and moment $M(t)$ and the histories of the displacements $w(R,t)$ and $g(R,t)$ that is complicated and computationally expensive to use in the design and engineering of vehicle suspensions. It was used instead to carry out numerical simulations in which displacement histories were specified and the corresponding forces and moments were calculated. The results were regarded as experimental data that were then used in a method developed in [55] to construct a force-displacement level constitutive equation for nonlinear viscoelastic response having the Pipkin–Rogers framework in (6.11)

$$\mathbf{M}(t) = \mathbf{R}(\mathbf{D}(t), 0) + \int_0^t \frac{\partial \mathbf{R}(\mathbf{D}(s), t-s)}{\partial (t-s)} \mathrm{d}s. \tag{19.3}$$

In (19.3), $\mathbf{M}(t) = [F(t), M(t)]$ and $\mathbf{D}(t) = [\hat{w}(t), \hat{g}(t)]$, $\hat{w}(t)$ and $\hat{g}(t)$ being the relative axial and rotational displacements, respectively. $\mathbf{R}(\mathbf{D}(t), t)$ is a displacement dependent relaxation function.

19.4. Other Applications

Elastomeric engine mounts are used in vehicles for noise and vibration isolation. Morman [56] considered a cylindrical engine mount that has rigid plates bonded to its end surfaces. Normal forces on the end plates compress the block while the lateral surface remains traction free. A simplified version of the Lianis constitutive equation (8.13) was used to model the material. With respect to cylindrical coordinates, the assumptions led to the following relation between the coordinates (r, θ, z) and (R, Θ, Z) in the current and reference configurations, respectively: $r = Rf(Z,t)$, $\theta = \Theta$, $z = g(Z,t)$, where $f(Z,t)^2 \partial g(Z,t)/\partial Z = 1$ on account of incompressibility. A partial differential-Volterra integral equation of form (19.2) was obtained for $f(Z,t)$, where now $\hat{D} = \hat{D}\left(f, \frac{\partial f}{\partial Z}, \frac{\partial^2 f}{\partial Z^2}\right)$ and $\hat{\hat{D}} = \hat{\hat{D}}\left(f, \frac{\partial f}{\partial Z}, \frac{\partial^2 f}{\partial Z^2}\right)$ denote partial differential operators on $f(Z,t)$. Boundary conditions were applied at the midsurface, $Z = 0$, and the top of the cylindrical block thereby making this a two point boundary value problem. The related problem of the non-uniform extension of a non-linear viscoelastic slab was treated independently in [57]. The Pipkin–Rogers constitutive equation (8.16) was used to model the material. The kinematical assumptions were equivalent to those in [56] and the formulation again resulted in a two point boundary value problem. Numerical methods of solution similar to those in the previous examples were provided.

Although there has been research into the development of constitutive equations that use the notion of the “strain” clock discussed in Section 10, there have been few studies in the literature that explore its implications. This is probably due to the fact that the computational effort required to use such constitutive equations is large. The constitutive equation in [28] was developed under the auspices of Sandia National Laboratory, which has extensive experimental and computational facilities. Most applications in this laboratory are project oriented and few results have been presented in the open literature. However, some work has appeared. In [58], a constitutive equation for a compressible nonlinear viscoelastic material was developed that incorporates a strain clock into the Pipkin–Rogers framework. The clock was chosen to have a simple dependence on shear and volume strains so that its essential features could be explored and yet be amenable to computation. In [58], a block was subjected to a homogeneous deformation consisting of shear superposed on triaxial extension. The dimensional, volume changes and shear response in the absence of normal tractions was studied. In [59], the constitutive equation was used to study circular shear of a cylinder. Owing to material compressibility, there can be radial motion of cylindrical surfaces as well as their rotation about the centerline. The influence of the “strain” clock on the moment-rotation relation, the radial variation of density and the possible growth of a shear boundary layer at the inner surface were explored.

The very time dependence of viscoelastic materials means that they dissipate energy. The influence of viscoelasticity on vibration damping was studied in [60–62]. A mass was attached to an elastomeric spring modeled by the constitutive equation for finite linear viscoelasticity. The equation of motion for the mass was an integro-differential equation, i.e. an equation in which the left-hand side of (11.1) was replaced by $\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2$. The integrand was such that the integro-differential could be replaced by a system of first order differential equations, which was then analyzed.

20. CONCLUDING COMMENTS

This article has presented an overview of the principal ingredients of nonlinear viscoelasticity, described constitutive equations of current interest, illustrated their application to several problems of technical relevance and discussed a variety of boundary value problems that have appeared in the technical literature. These problems have been formulated with a semi-inverse method similar to that used in nonlinear elasticity. In nonlinear elasticity, the spatial dependence of the deformation is represented by an assumed expression containing parameters that determine the magnitude of the deformation. In nonlinear viscoelasticity, the deformation is embedded in a motion by letting the parameters in this expression be functions of time. The stretch histories in Sections 13–15, the simple shear history of Section 16, the deformations of Section 17 and the other examples discussed in Section 19 were formulated in this manner. Thus, as observed in [45] and [51], boundary value problems in nonlinear elasticity suggest corresponding problems in nonlinear viscoelasticity. These problems, as in the examples cited here, lead to partial differential-Volterra equations that can be solved by the methods in the references mentioned in Section 19.

The above comments should not be construed as suggesting that nonlinear viscoelasticity is just a straightforward extension of nonlinear elasticity. Nonlinear viscoelasticity incorporates the same interesting phenomena as nonlinear elasticity. However, the time dependent behavior of nonlinear viscoelastic solids adds a layer of new and interesting phenomena to be investigated.

Two classes of nonlinear single integral constitutive equations have been presented, one for finite linear viscoelasticity and the other for the Pipkin–Rogers theory. The Lianis model, in the former case, was the result of an extensive experimental program for a particular kind of rubber. The quasi-linear viscoelastic model, a special version of the latter case, is extensively used to represent the mechanical response of biological tissue. These models were developed using a limited set of deformation histories. Morman [3] has suggested that it may be impractical to develop a constitutive equation for all deformation histories. Instead, special classes of constitutive equations should be developed for special purposes. For example, the development of a constitutive equations for small amplitude oscillations about a finite fixed deformation was the motivation behinds the work in [36] and [37]. The influence of the underlying finite deformation on the superposed motion is an interesting topic for investigation.

There appear to have been few studies of the response of nonlinear viscoelastic solids under a broader set of conditions that include multi-axial deformations, for both deformation and stress control conditions and with unloading. Moreover, isotropic materials have received most of the attention. There is a paucity of studies that explore the consequences of anisotropy. When the deformation is homogeneous, it is useful to have an estimate of characteristic times for processes such as stress relaxation, creep, and recovery on unloading as well as an understanding of how these times depend on the deformation. If this is the case, in a non-homogeneous deformation, the time for the deformation or stress field to evolve may vary significantly throughout a body. This also appears to be an unexplored area.

An interesting phenomenon occurred in the study of the inflation of a spherical viscoelastic membrane under internal pressure. It was shown that there might be a time when the solution for the deformation branches into several solutions. This event depends on the pressure history and the material parameters. There has not yet been a study of this phenomenon for non-homogeneous deformations. Such a study would determine the conditions for a branching time to exist, the branching time, the solution branches, and a criterion for the selection of the appropriate branch followed by the material. An important related topic for investigation is time dependent stability of viscoelastic bodies.

Another important and relatively unexplored consequence of the time dependence of viscoelastic materials is that they dissipate energy and heat up as a result of thermo-mechanical coupling. Owing to poor heat conduction, there may be regions of significant temperature rise in a viscoelastic component as it undergoes cyclic loading. Although, this has been studied in the context of linear viscoelasticity, the influence of material nonlinearity on the process has received little attention.

Biological tissues generally exhibit nonlinear viscoelastic behavior. Indeed, the constitutive equation for quasi-linear viscoelasticity has been used to model the response of a variety of such materials. The topics presented in previous sections provide tools and examples that can be applied to the study of the mechanics of bodies composed of biological tissue. For example, fibrous tissue in blood vessels is often regarded as transversely isotropic or orthotropic. The Pipkin–Rogers constitutive equation in Section 9 provides a framework for combining these material symmetries with nonlinear viscoelasticity. The mechanics of the combined extension, inflation and torsion of cylinders of anisotropic materials can then be treated as in Section 18. Another important topic is that of the influence of viscoelasticity on cell mechanics. The literature on viscoelastic membranes described in Section 19 provides the foundation for such research.

The preceding paragraphs describe just a few topics for further investigation that are inherently associated with the time dependence of nonlinear viscoelastic solids. Many other topics can be anticipated as a result of the development of polymers that couple viscoelastic phenomena with other physical fields, i.e. fluid diffusion or polymers that are optically and electrically active. With the overview of nonlinear viscoelastic solids provided in this article, it is hoped that one can rapidly become comfortable in the subject and have the background to begin to explore these new topics.

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